Rendering = Integrals

Mathematical Aspects and Algorithms underlying PBR

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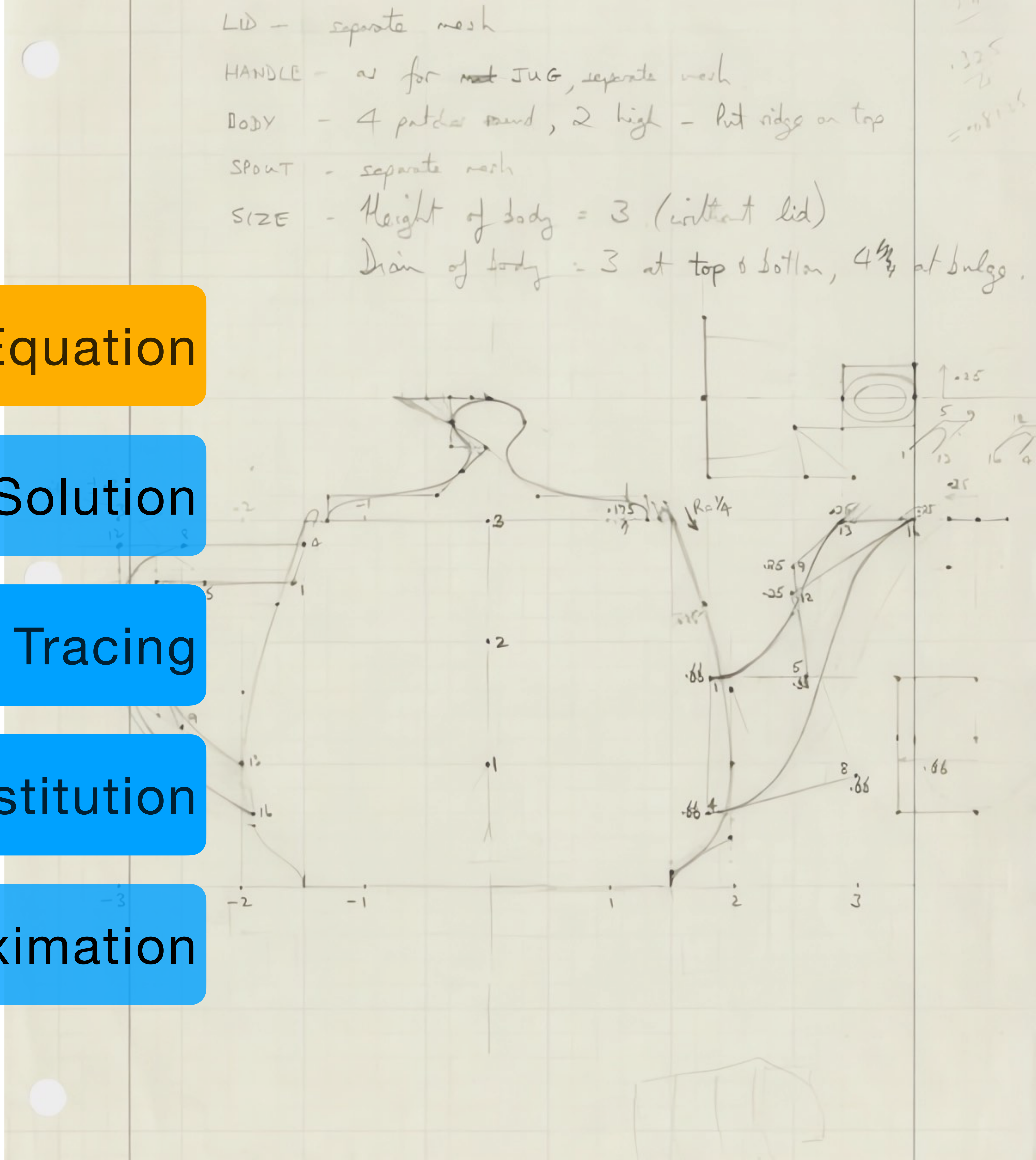
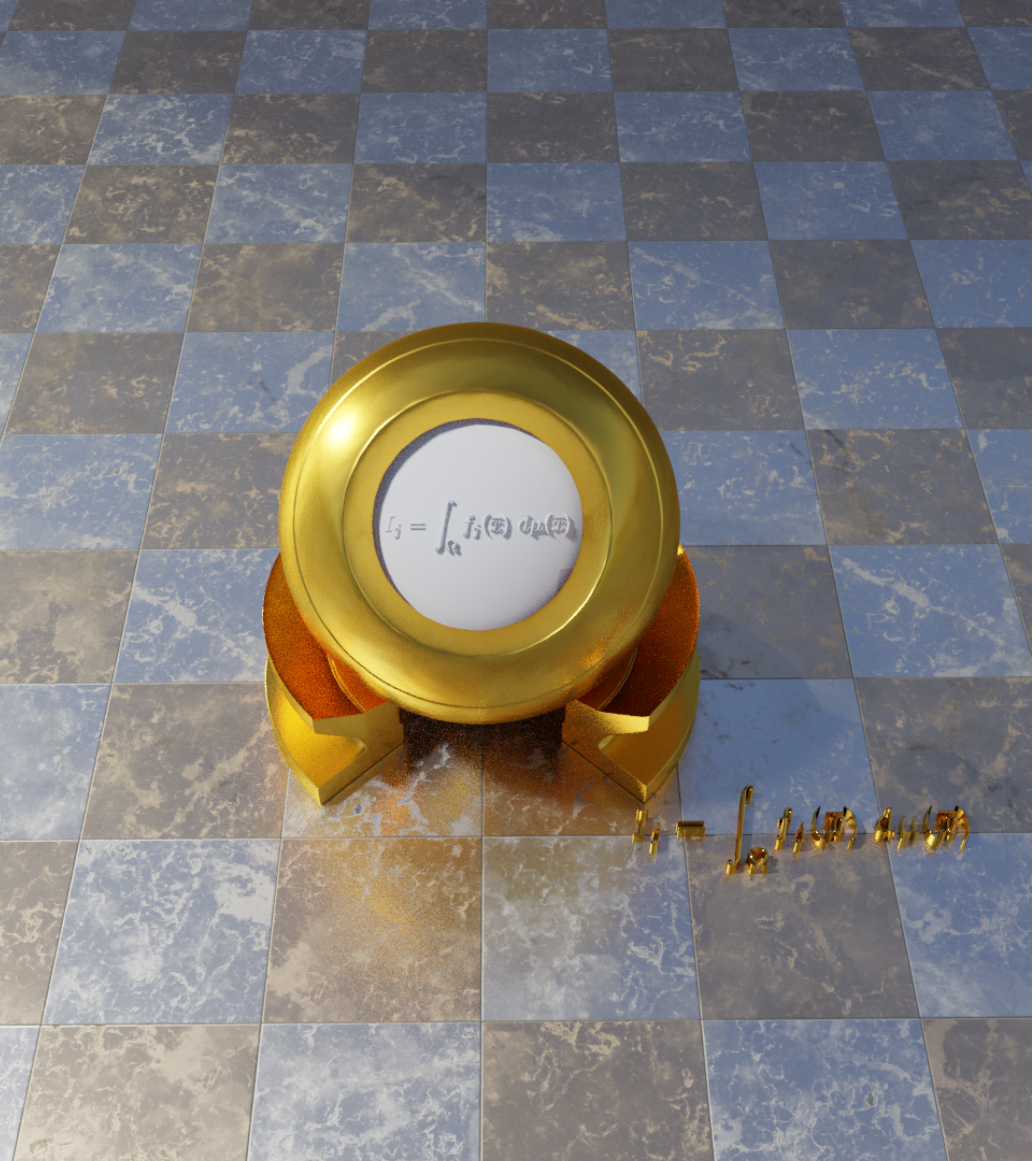
Overview

- The Rendering Equation
- Analytic Solution
- Monte Carlo Path Tracing
- Integration by Substitution
- Split-Sum Approximation





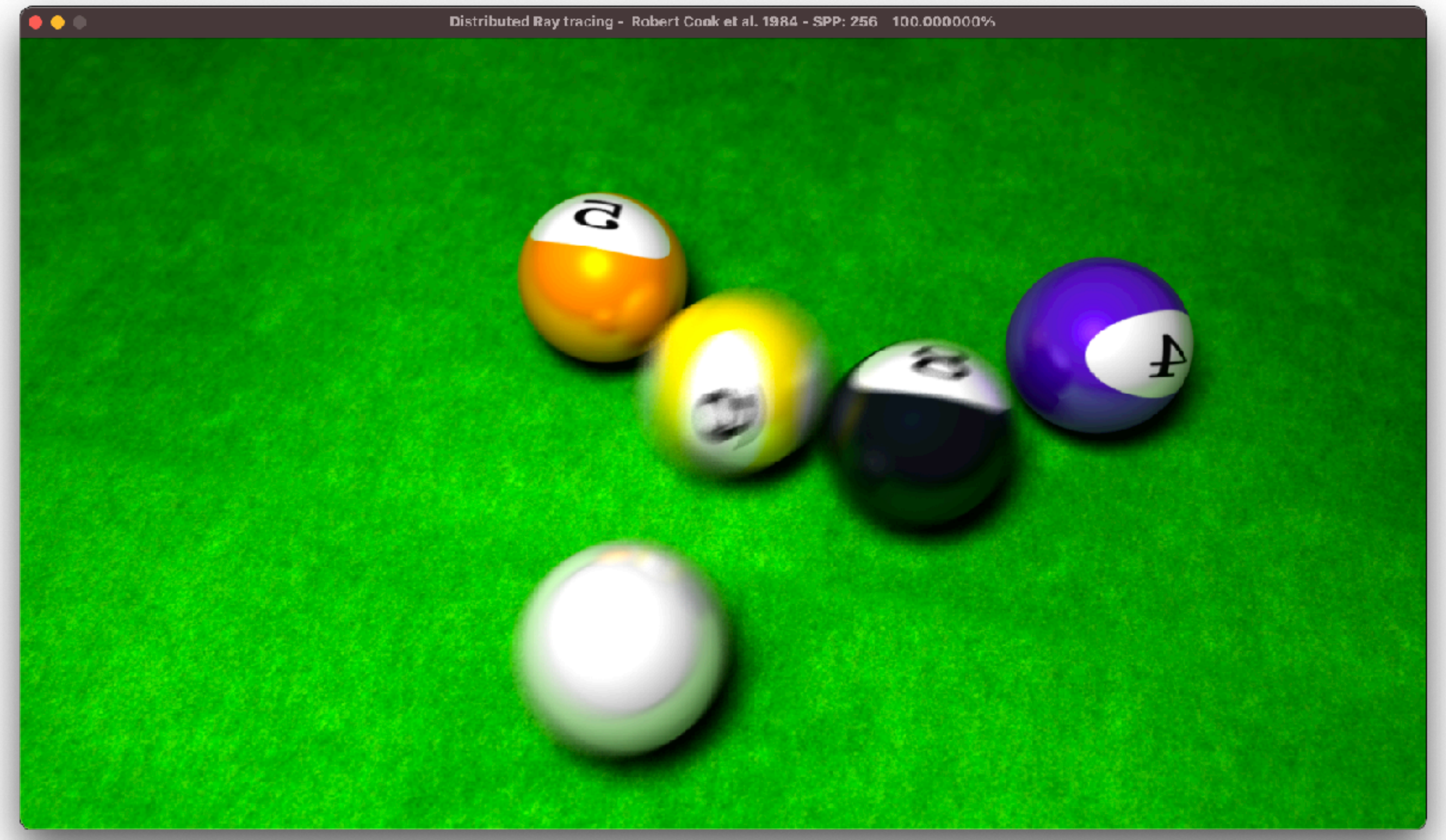
The Rendering Equation

渲染方程的发展与基础知识

渲染方程的萌芽

Distributed Ray Tracing, Robert Cook et al., January 1984

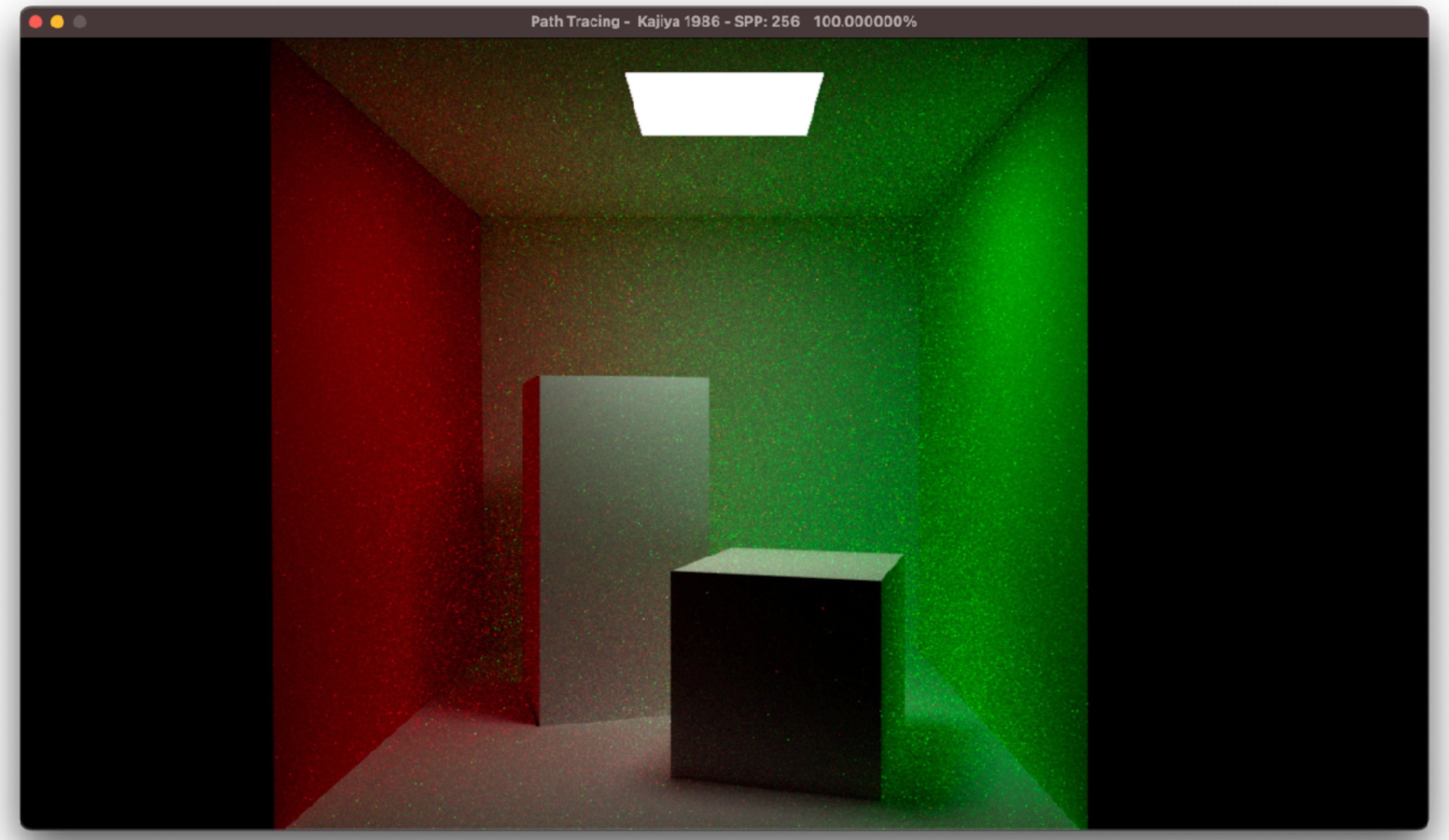
$$I(\phi_r, \theta_r) = \int_{\phi_i} \int_{\theta_i} L(\phi_i, \theta_i) R(\phi_i, \theta_i, \phi_r, \theta_r) \mathrm{d}\phi_i \mathrm{d}\theta_i$$



渲染方程的正式提出

The rendering equation, James Kajiya, August 1986

$$L_r(\omega_o) = L_e(\omega_o) + \int_{\Omega} f(\omega_i, \omega_o) L_i(\omega_i)(n \cdot \omega_i) \mathrm{d}\omega_i$$



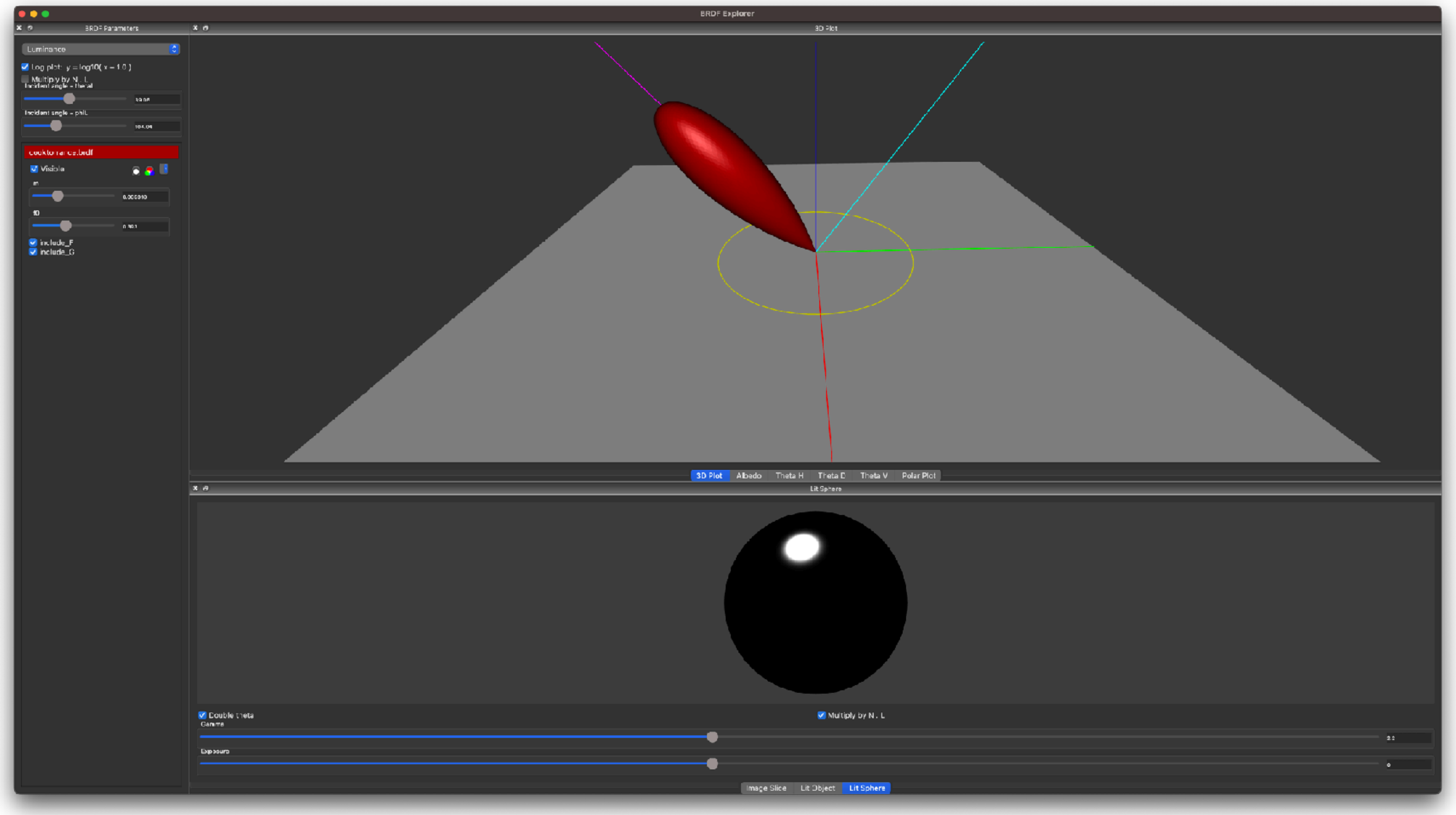
渲染的正确性

The focus is on being correct, not just producing a pretty image.

It's easy to produce a pretty image which has a number of subtle bugs.

关于BRDF

- Reciprocity: $f(\mathbf{l}, \mathbf{v}) = f(\mathbf{v}, \mathbf{l})$
- Energy Conservation: $\forall \mathbf{l}, \int_{\Omega} f(\mathbf{l}, \mathbf{v})(\mathbf{n} \cdot \mathbf{v}) d\omega_o \leq 1$



The Disney BRDF Explorer

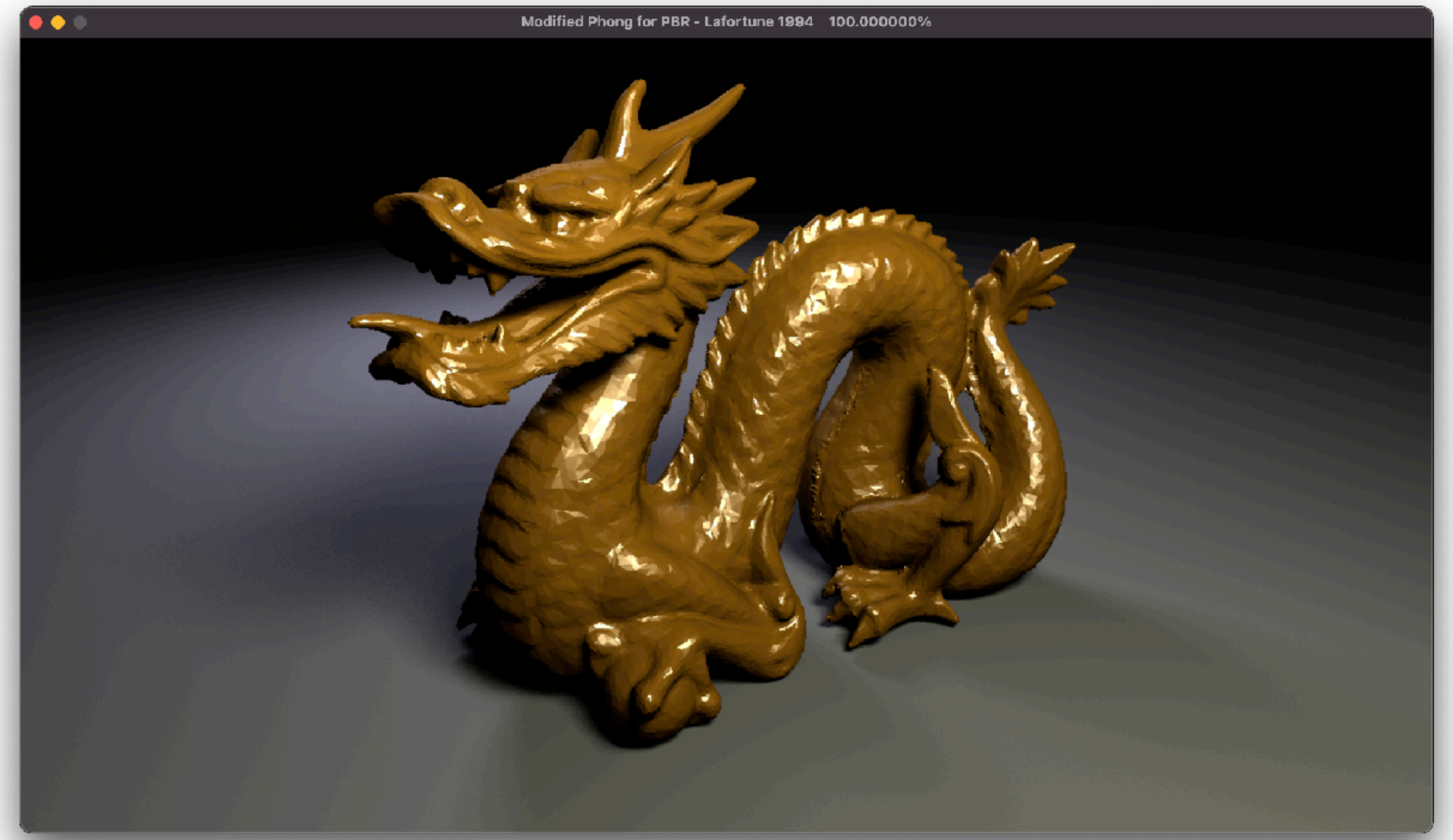
提问: Phong 模型是不是一个 BRDF?

关于BRDF

Using the modified phong reflectance model for physically based rendering

Eric Lafortune et al., November 1994

$$f(\omega_i, \omega_o) = \frac{k_d}{\pi} + k_s \frac{s+2}{2\pi} (r \cdot \omega_i)^s$$



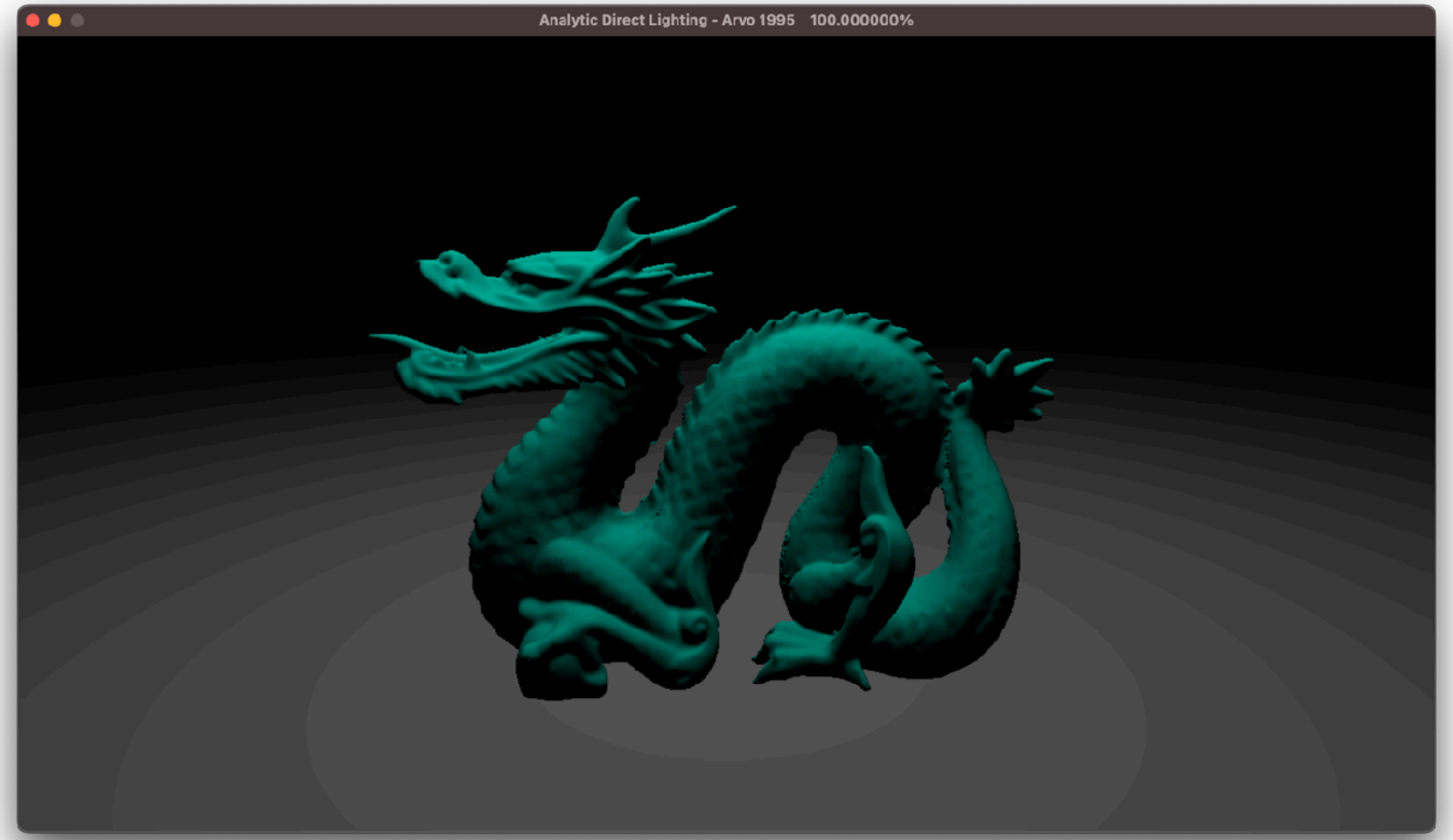
Analytic Solution

在特定条件下的解析解

渲染方程的解析解

Analytic Methods for Simulated Light Transport

James Richard Arvo, January 1995



渲染方程的解析解

Analytic Methods for Simulated Light Transport

James Richard Arvo, January 1995

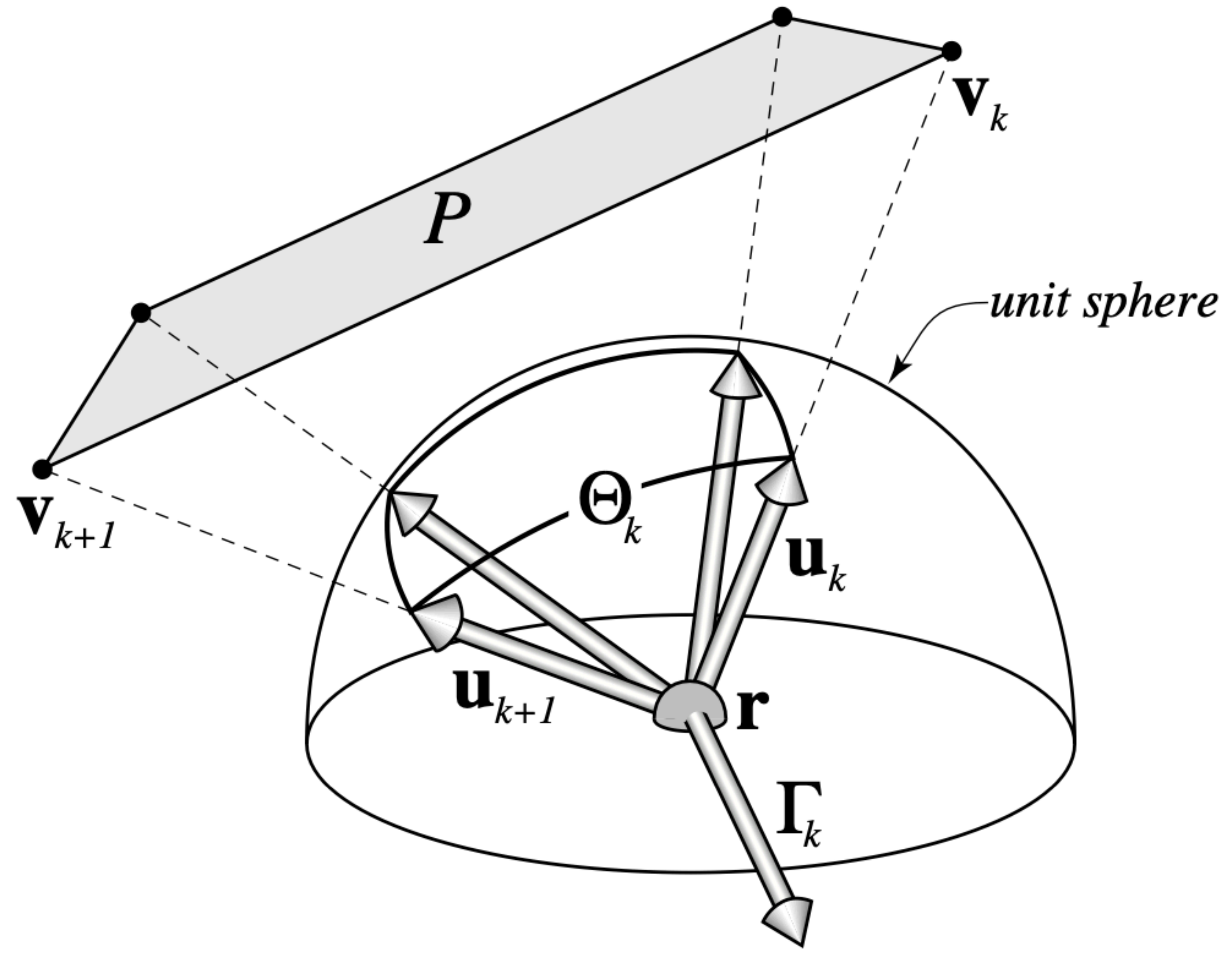
- *Lambertian BRDF 是一个常量，可以提到积分外面来*

$$f = \frac{k_d}{\pi}$$

$$L_d(\omega_o) = f \int_{\Omega_P} L_i(n \cdot \omega_i) d\omega_i$$

渲染方程的解析解

- 假设面光源的能量是均匀的
- 渲染方程的求解转化为：多变形在单位球面上的投影面积



$$L_d(\omega_o) = \frac{k_d}{\pi} L_i * (\Phi(r) \cdot n(r))$$

$$\Phi(r) = \frac{1}{2} \sum_{i=1}^{n} \Theta_i(r)\Gamma_i(r),$$

$$\Theta_k(r) = cos^{-1}\left(\frac{v_k - r}{\|v_k - r\|} \cdot \frac{v_{k+1} - r}{\|v_{k+1} - r\|}\right) \qquad \Gamma_k(r) = \frac{(v_k - r) \times (v_{k+1} - r)}{\|(v_k - r) \times (v_{k+1} - r)\|}$$

Monte Carlo Path Tracing

Gold Standard for Rendering

Monte Carlo Integration

Monte Carlo Integration

Definite integral $I(f) \equiv \int_0^1 f(x)\,dx$

Expectation of f $E[f] \equiv \int_0^1 f(x)p(x)\,dx$

Random variables $X_i \sim p(x)$

$Y_i = f(X_i)$

Estimator $F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$

Monte Carlo Integration

Unbiased Estimator

$$E[F_N] = I(f)$$

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i] = \frac{1}{N}\sum_{i=1}^{N} E[f(X_i)]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)p(x)\,dx$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)dx$$

$$= \int_0^1 f(x)\,dx$$

Properties

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$

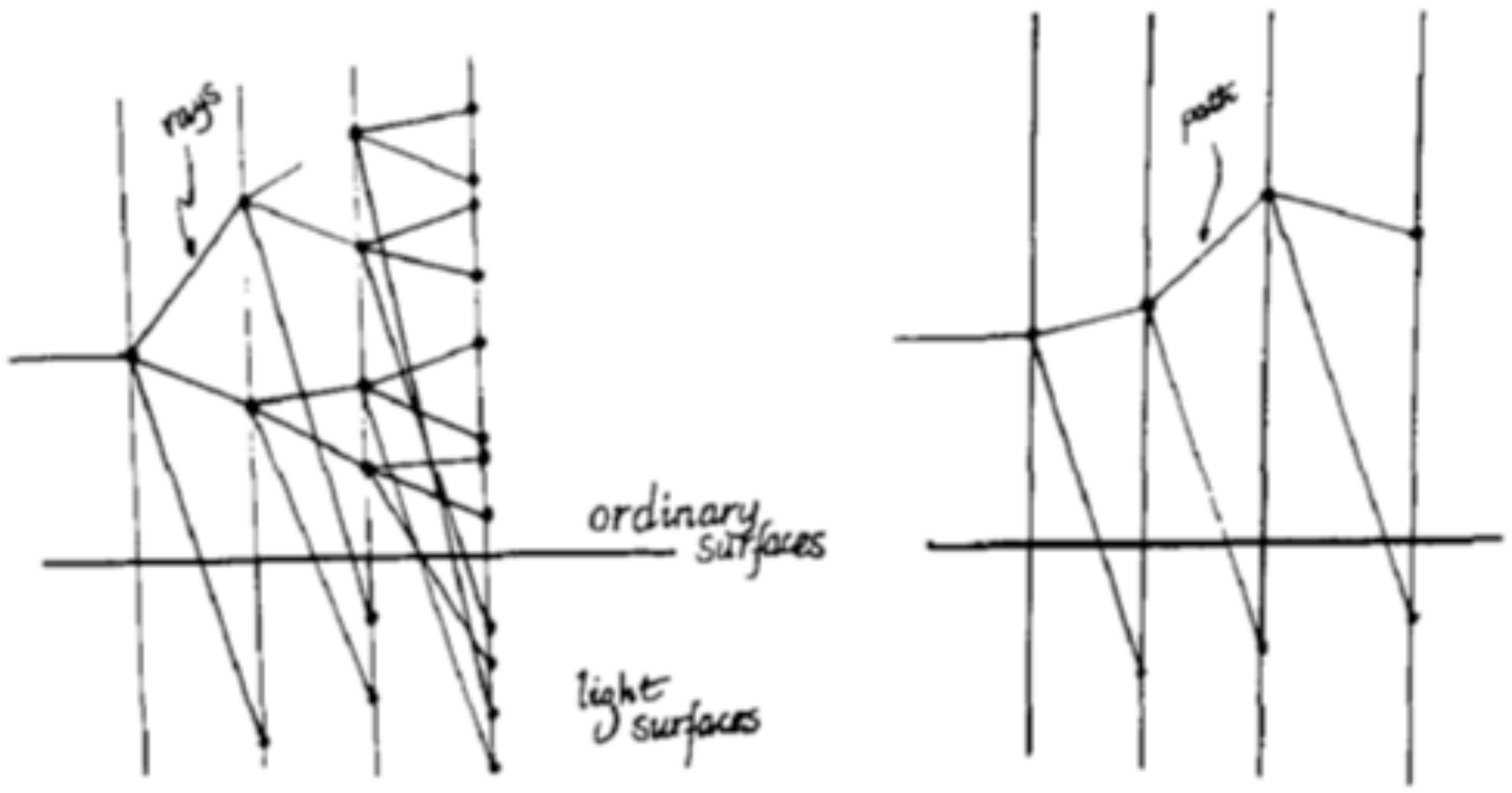
$$E[aY] = aE[Y]$$

Linearity of Expectation

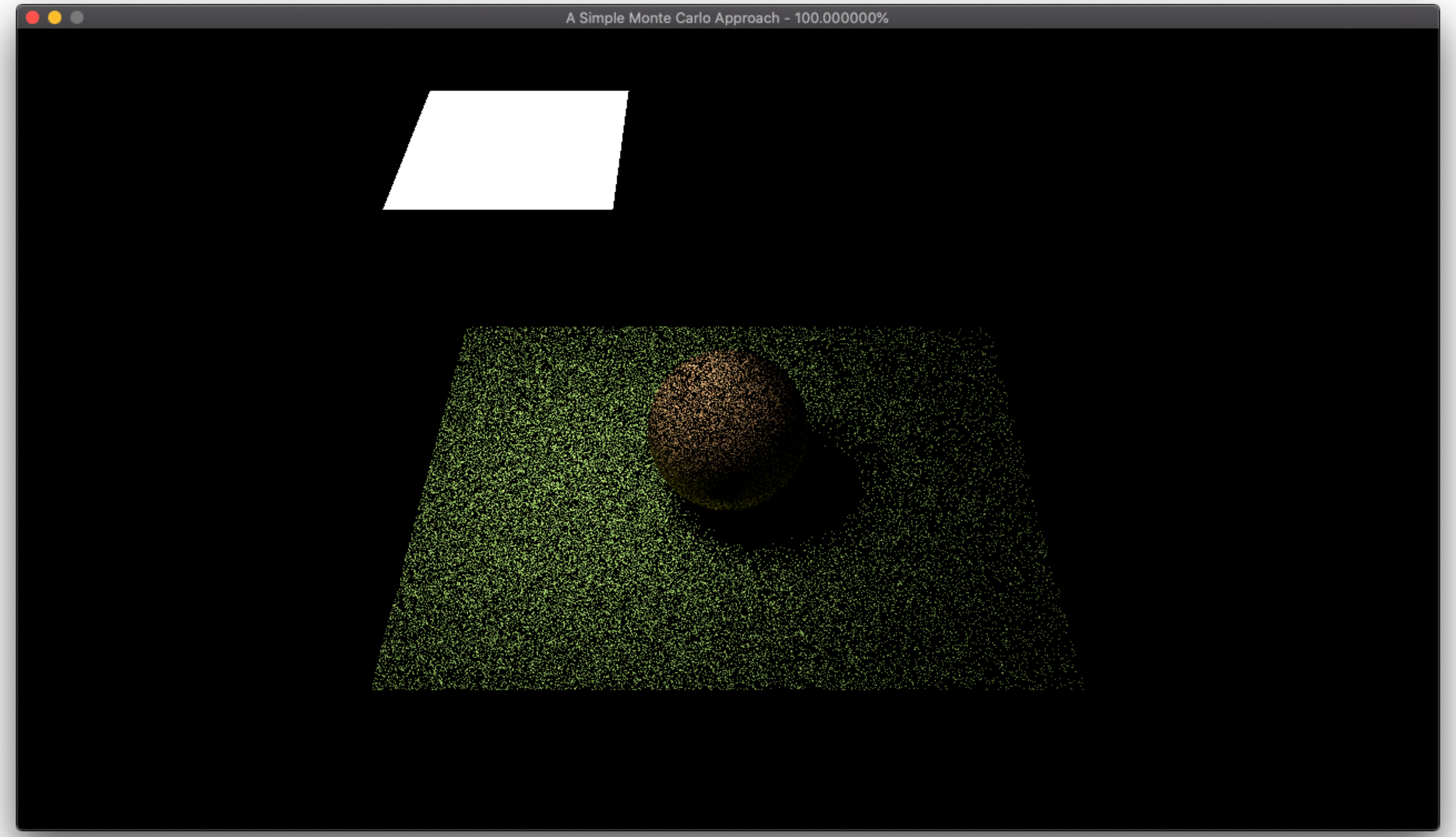
Assume uniform probability distribution for now

CS348B Lecture 6 Pat Hanrahan, Spring 2005

原始的Path Tracing



Path Tracing



- 核心的问题是：采样效率低

BRDF Importance Sampling

Continuous Probability Distributions

PDF $p(x)$

$$p(x) \geq 0$$

CDF $P(x)$

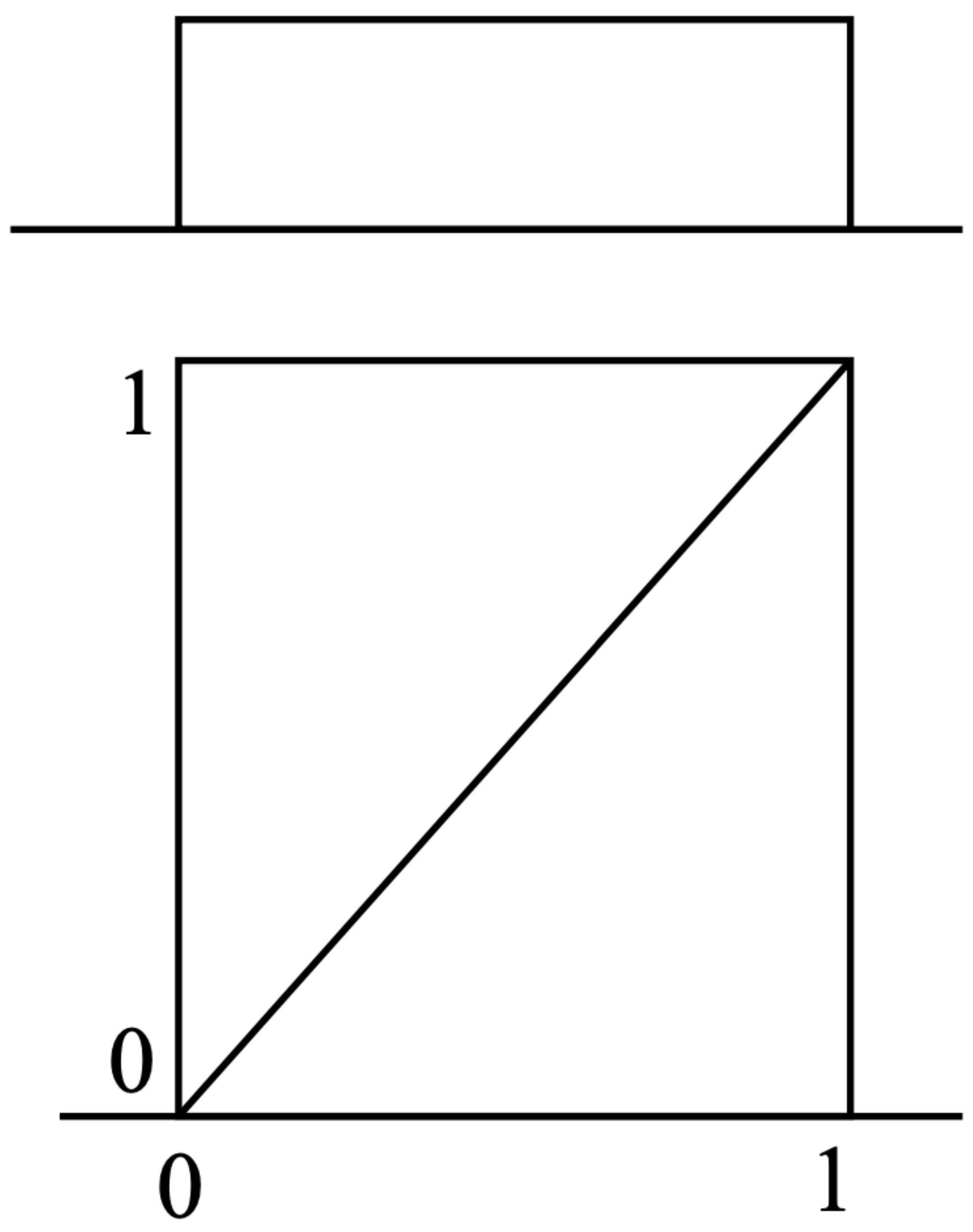
$$P(x) = \int_0^x p(x)dx$$

$$P(x) = \Pr(X < x) \qquad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$

Uniform



BRDF Importance Sampling

Sampling Continuous Distributions

Cumulative probability distribution function

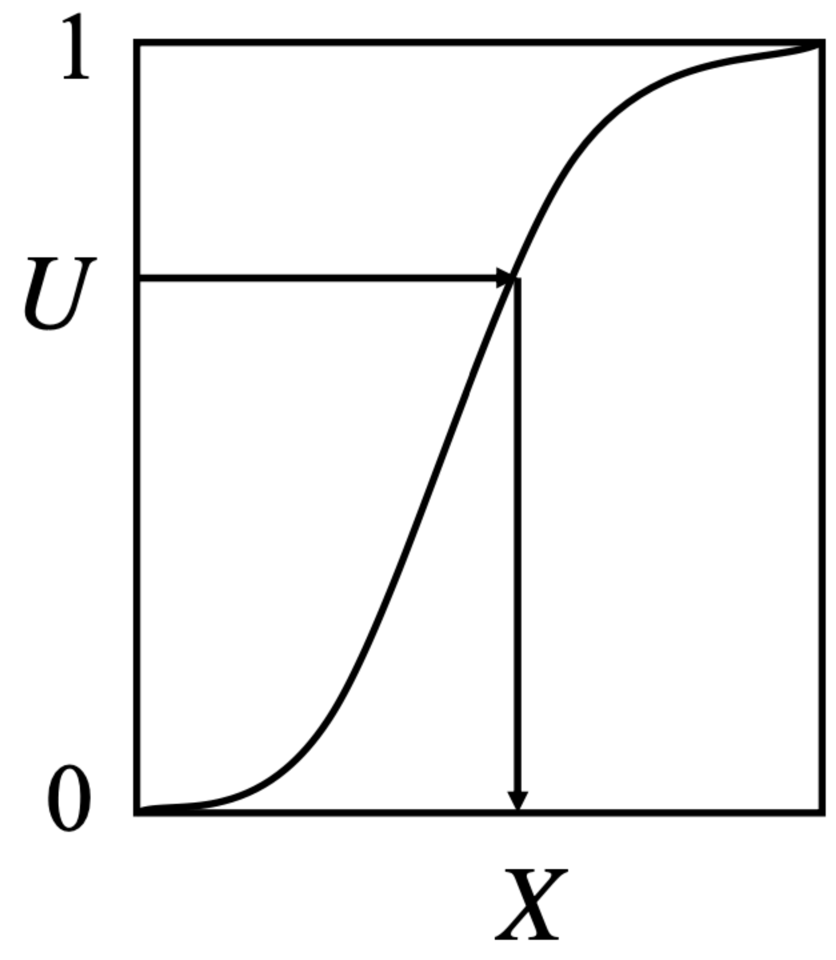
$$P(x) = \Pr(X < x)$$

Construction of samples

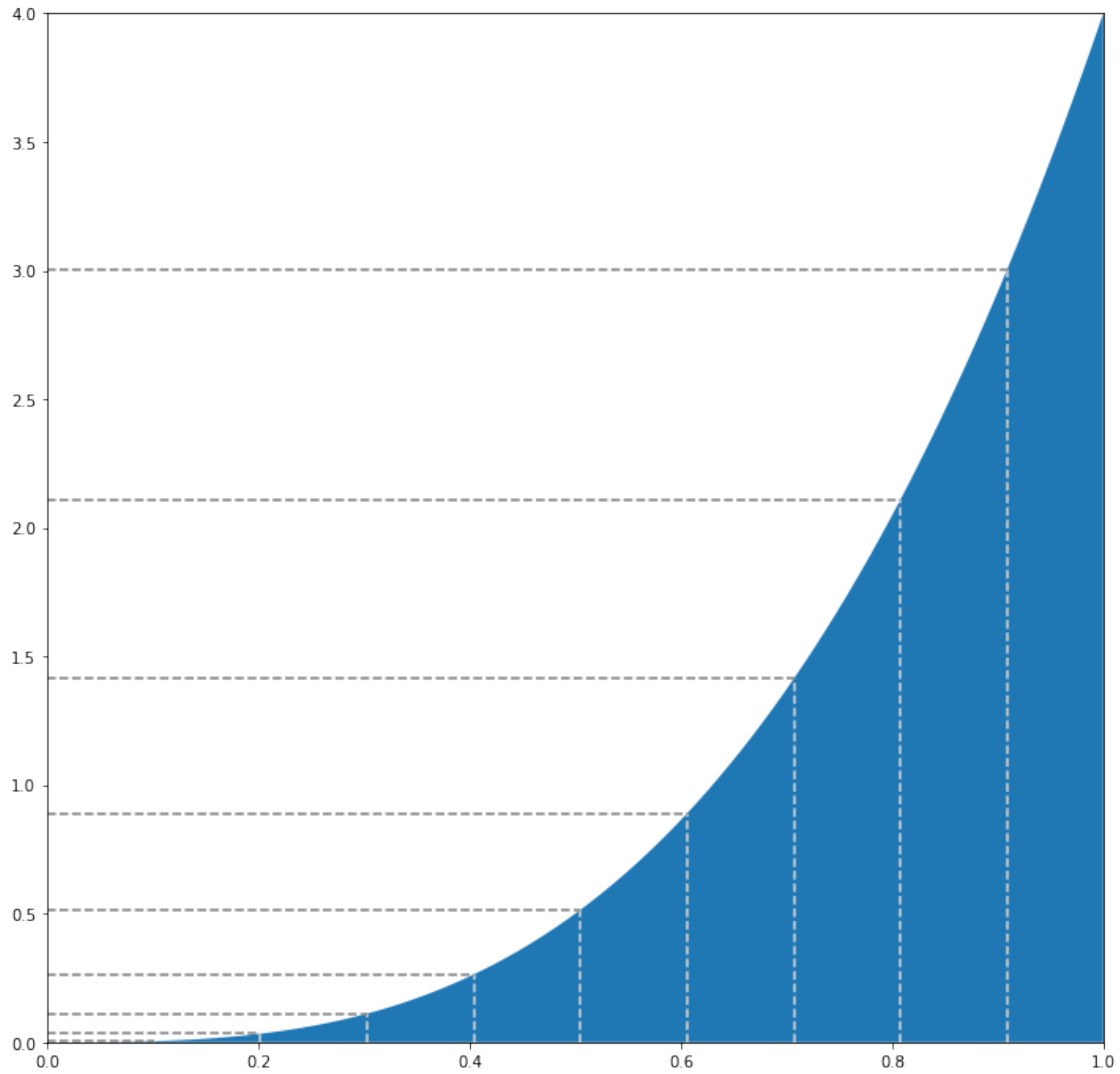
Solve for $X = P^{-1}(U)$

Must know:

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$



BRDF Importance Sampling



The Inversion Method

BRDF Importance Sampling

Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

Trick

$$Y = \max(U_1, U_2, \cdots, U_n, U_{n+1})$$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U < x) = x^{n+1}$$

Monte Carlo Direct Lighting

Monte Carlo techniques for direct lighting calculations

Peter Shirley et.al,January 1996

- The integral is now over the area of the light source polygon P

$$L_d(\omega_o) = L_i \int_P f(\omega_i, \omega_o)(n \cdot \omega_i) \frac{n_{light} \cdot \omega_i}{R^2} \mathrm{d}A$$

$$\mathrm{d}\omega_i = \frac{n_{light} \cdot \omega_i}{R^2} \mathrm{d}A$$

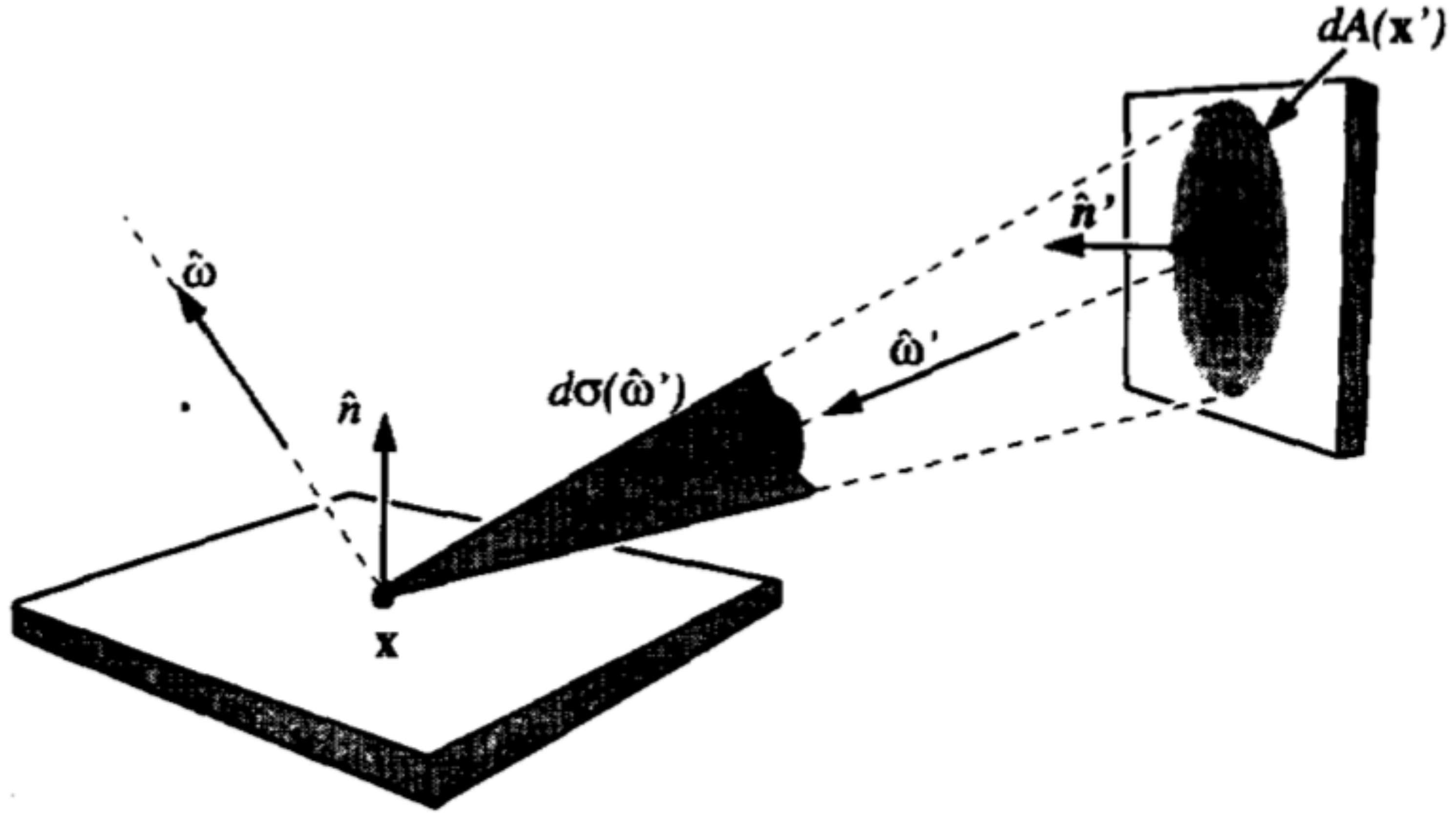


Fig. 1. Geometry for rendering equation.

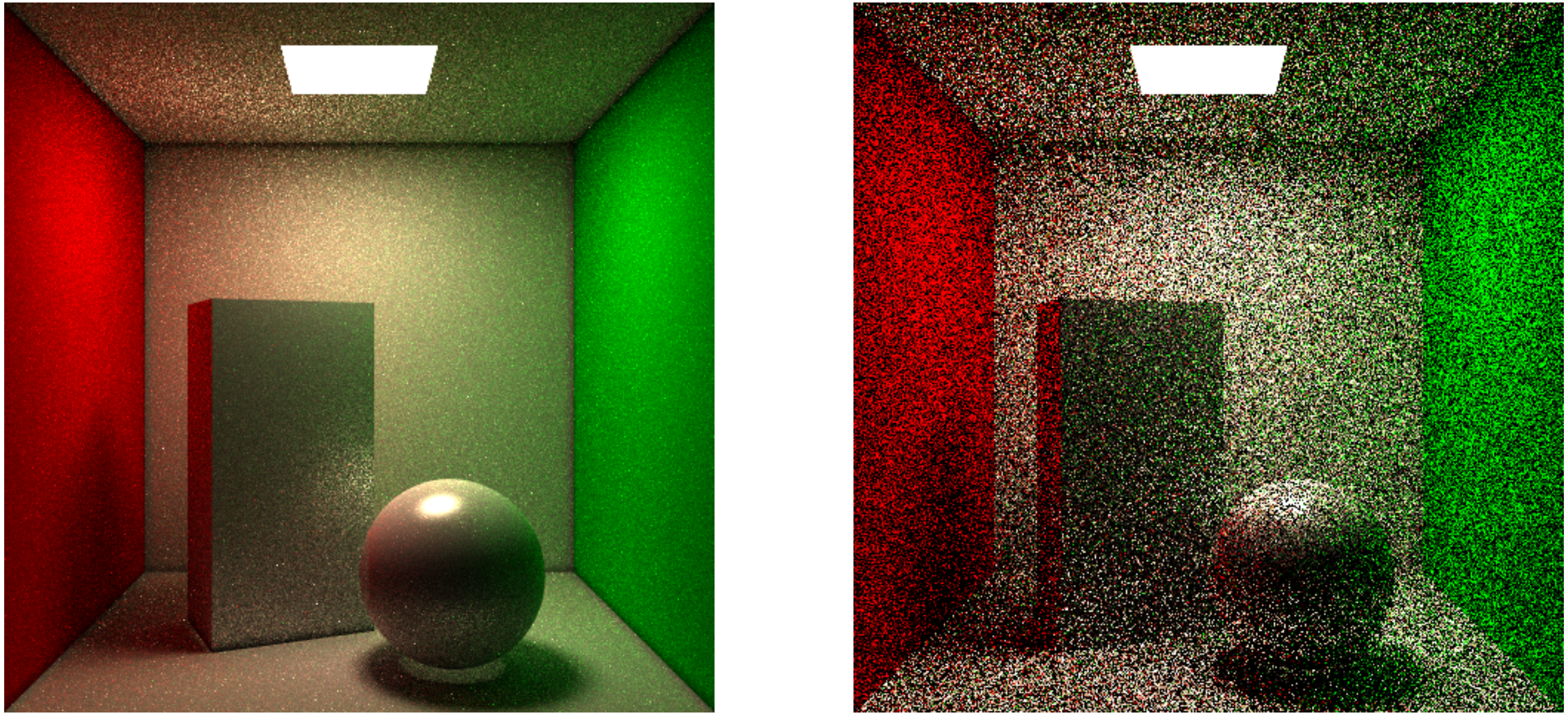
Next Event Estimation

- Direct/Indirect lighting separation

$$L_i(x, \omega_i) = L_{dir}(x, \omega_i) + L_{ind}(x, \omega_i)$$

$$L_r(x, \omega_o) = L_e(x, \omega_o) + \int_{\Omega} f(\omega_i, \omega_o) L_{dir}(x, \omega_i)(n \cdot \omega_i) \mathrm{d}\omega_i + \int_{\Omega} f(\omega_i, \omega_o) L_{ind}(x, \omega_i)(n \cdot \omega_i) \mathrm{d}\omega_i$$

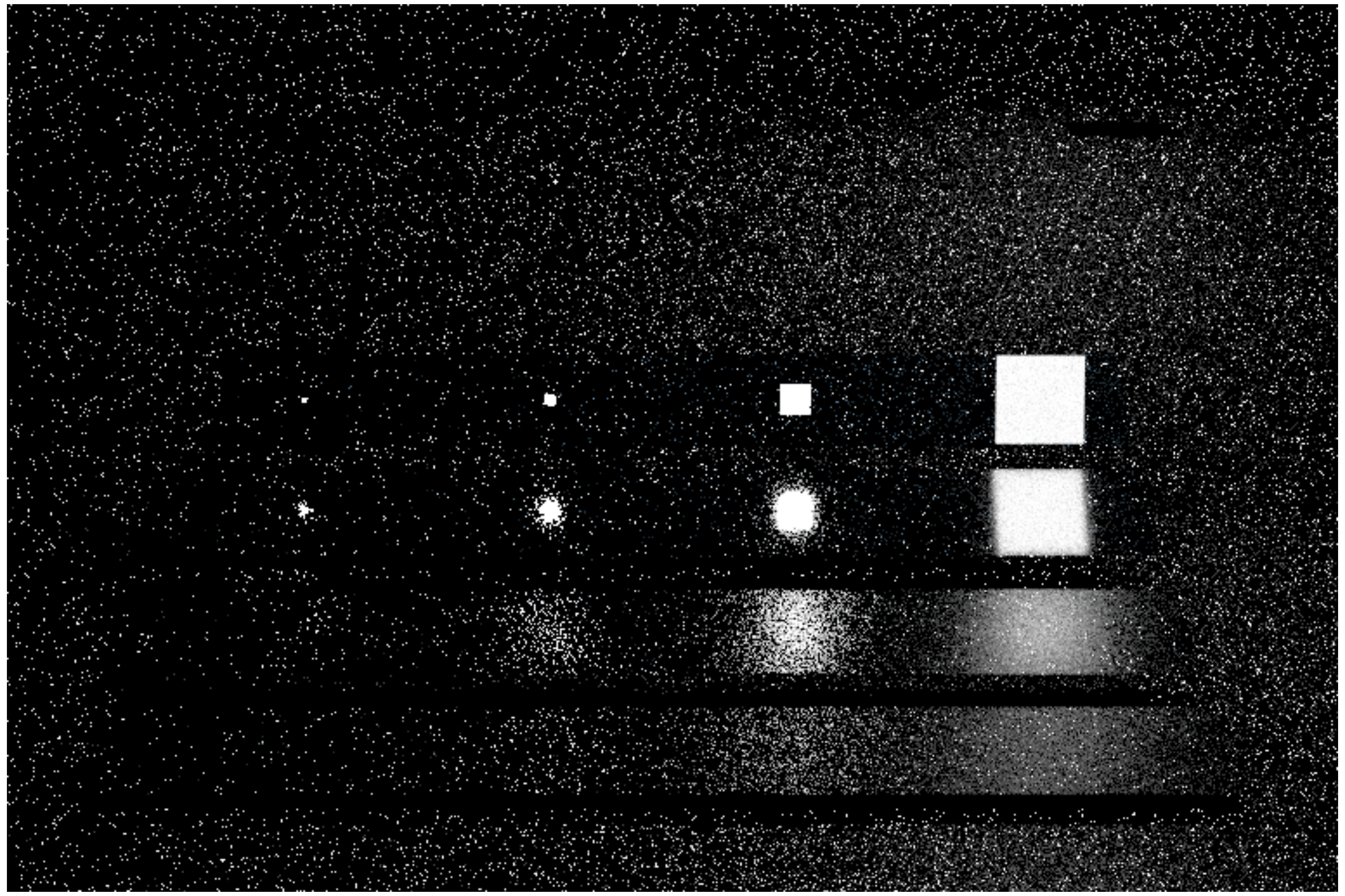
Monte Carlo Direct Lighting



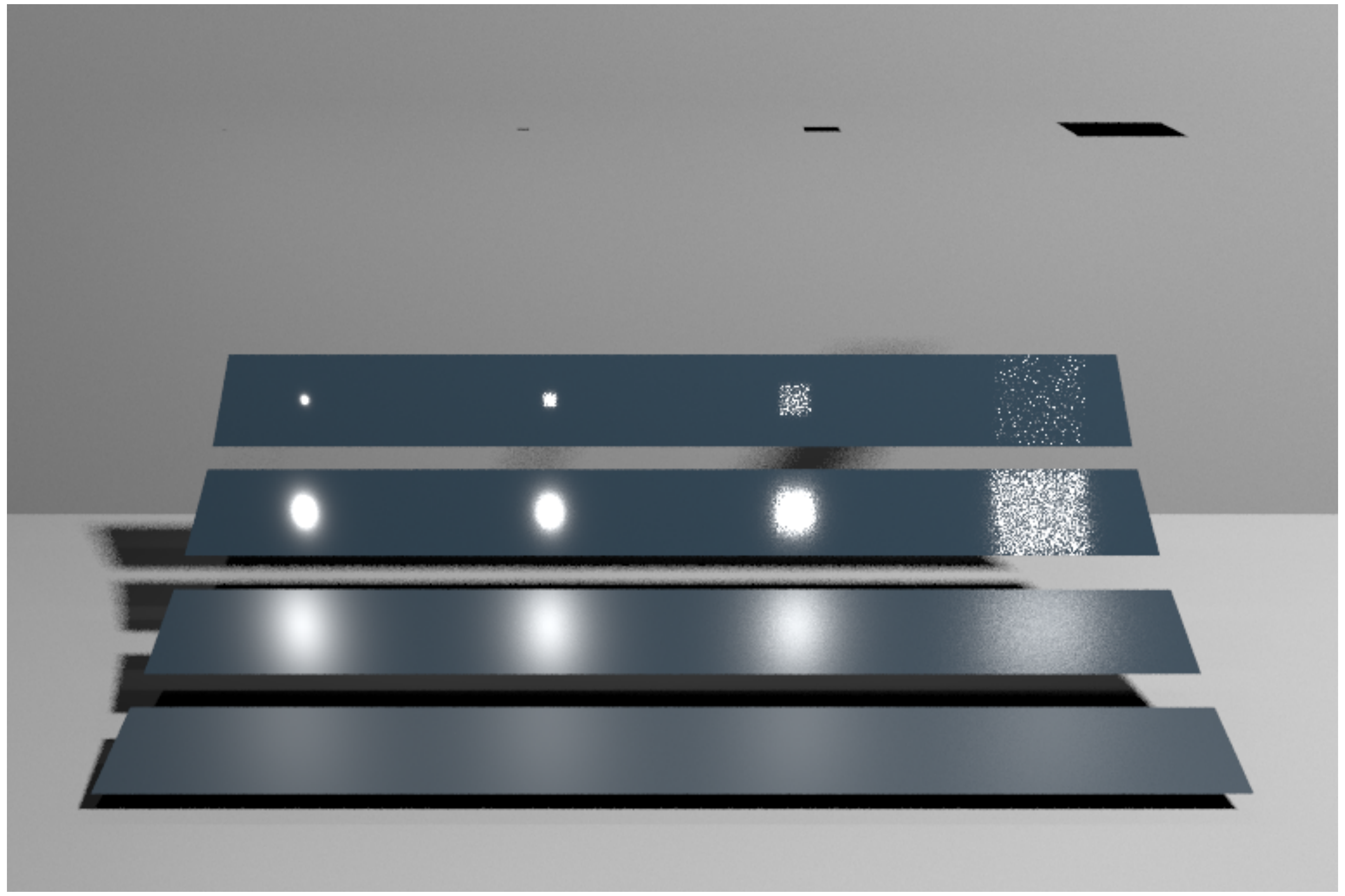
使用 NEE 之后的比较

理一下思路

- 对渲染方程的两个核心函数进行重要性采样



BRDF importance sampling



Next event estimation

Can we combine BRDF and Light(NEE) sampling?

Multiple Importance Sampling

Optimally Combining Sampling Techniques for Monte Carlo Rendering

Eric Veach et.al, September 1995

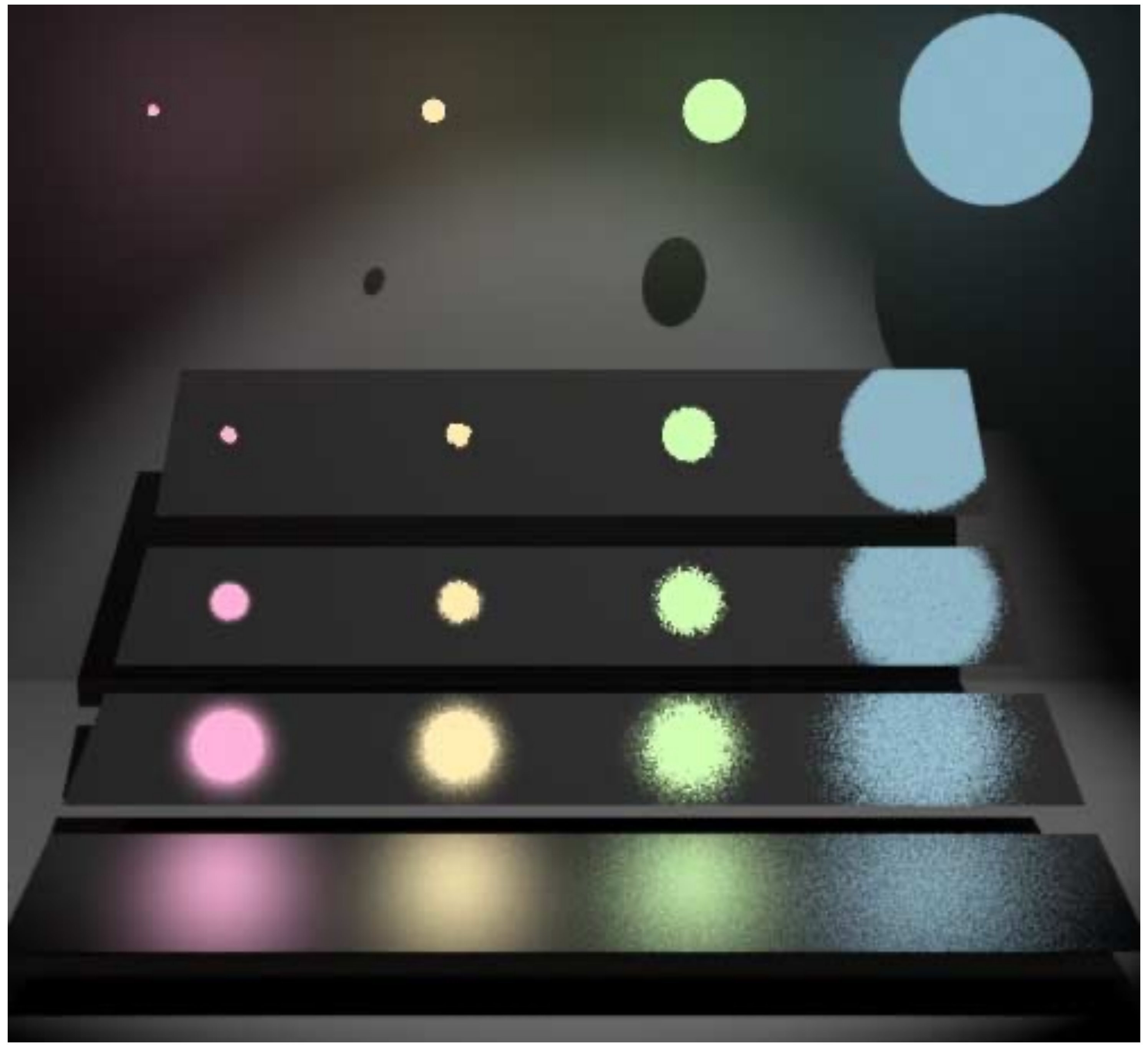
- Unbiased

$$\int_X f(x)dx \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(x_{ij}) \frac{f(x_{ij})}{pdf_i(x_{ij})}$$

- Power heuristics

$$w_i(\omega) = \frac{pdf_i^{\beta}(\omega)}{\sum_{k=1}^{N} pdf_k^{\beta}(\omega)}$$

Multiple Importance Sampling



Combination of BRDF and Light(NEE) sampling

Academy Awards for Scientific and Technical Achievement

Arnold 渲染器

Arnold is an advanced Monte Carlo ray tracing renderer

Marcos Fajardo, SIGGRAPH 2001

2001 SIGGRAPH

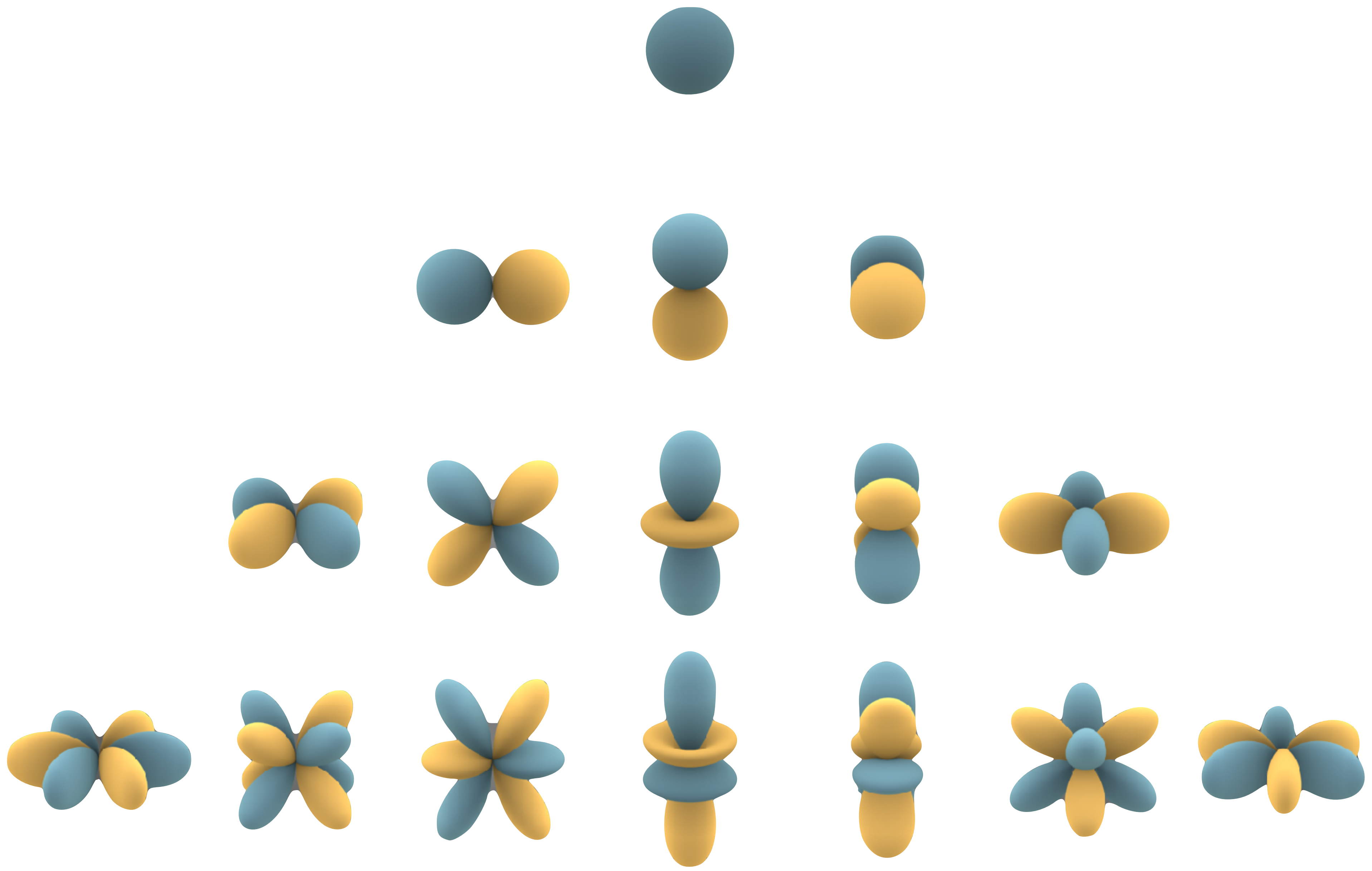
2017 MARVEL

Integration by Substitution

积分替换法求解

积分替换法

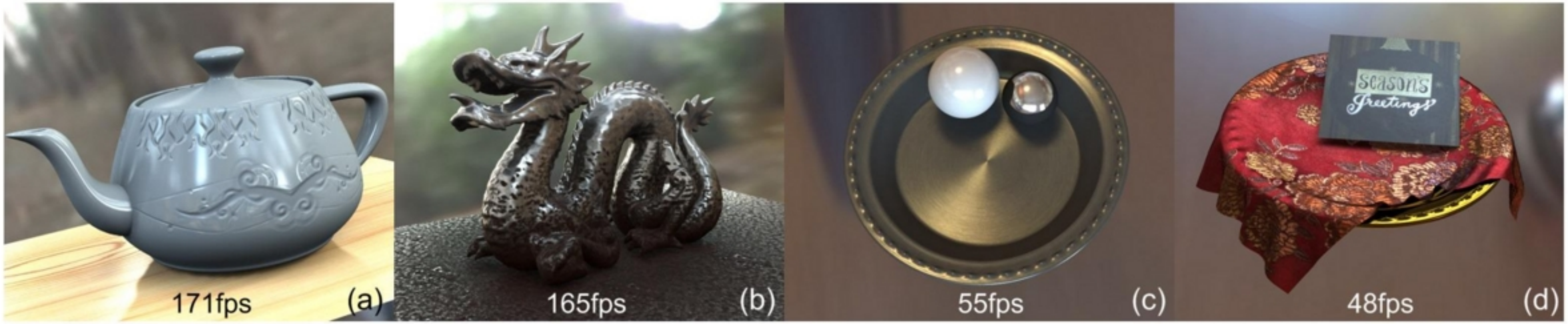
- 使用“简单函数”替换被积函数，利用新函数的一些特殊性质，来获得解析解
- Spherical Gaussians
- Linearly Transformed Cosines
- Spherical Harmonics



Spherical Gaussians

All-Frequency Rendering of Dynamic, Spatially-Varying Reflectance

王嘉平, et al., January, 2007



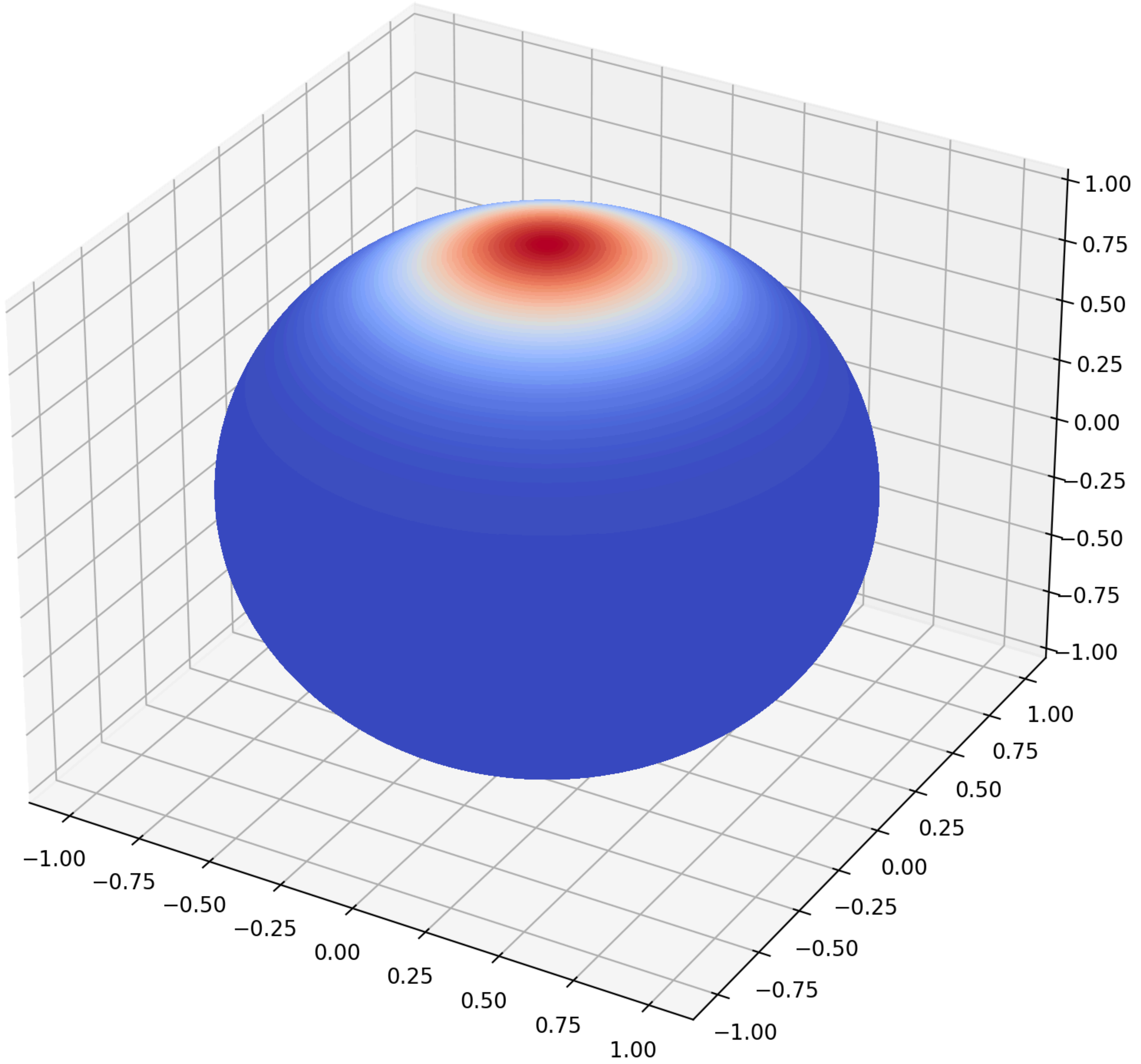
Spherical Gaussians

- Definition

$$G(\mathbf{v};\mathbf{p},\lambda,\mu)=\mu e^{\lambda(\mathbf{v}\cdot\mathbf{p}-1)}$$

- The Inner Product

$$G_1\cdot G_2=\int_{\mathbb{S}^2}G_1(\mathbf{v})G_2(\mathbf{v})\mathrm{d}\mathbf{v}=\frac{4\pi\mu_1\mu_2}{e^{\lambda_1+\lambda_2}}\frac{\sinh\left(d_m\right)}{d_m}$$



Spherical Gaussians

- BRDF Decomposition

$$R(\mathbf{o}) = k_d R_d + k_S R_S(\mathbf{o})$$

$$R_d = \int_\Omega L(\mathbf{i})\rho_D(\mathbf{o}, \mathbf{i}) \max(0, \mathbf{i} \cdot \mathbf{n}) \mathrm{d}\omega$$

$$R_S(\mathbf{o}) = \int_\Omega L(\mathbf{i})\rho_S(\mathbf{o}, \mathbf{i}) \max(0, \mathbf{i} \cdot \mathbf{n}) \mathrm{d}\omega$$

Spherical Gaussians

- Subsurface Scattering 只发生在 Diffuse 部分

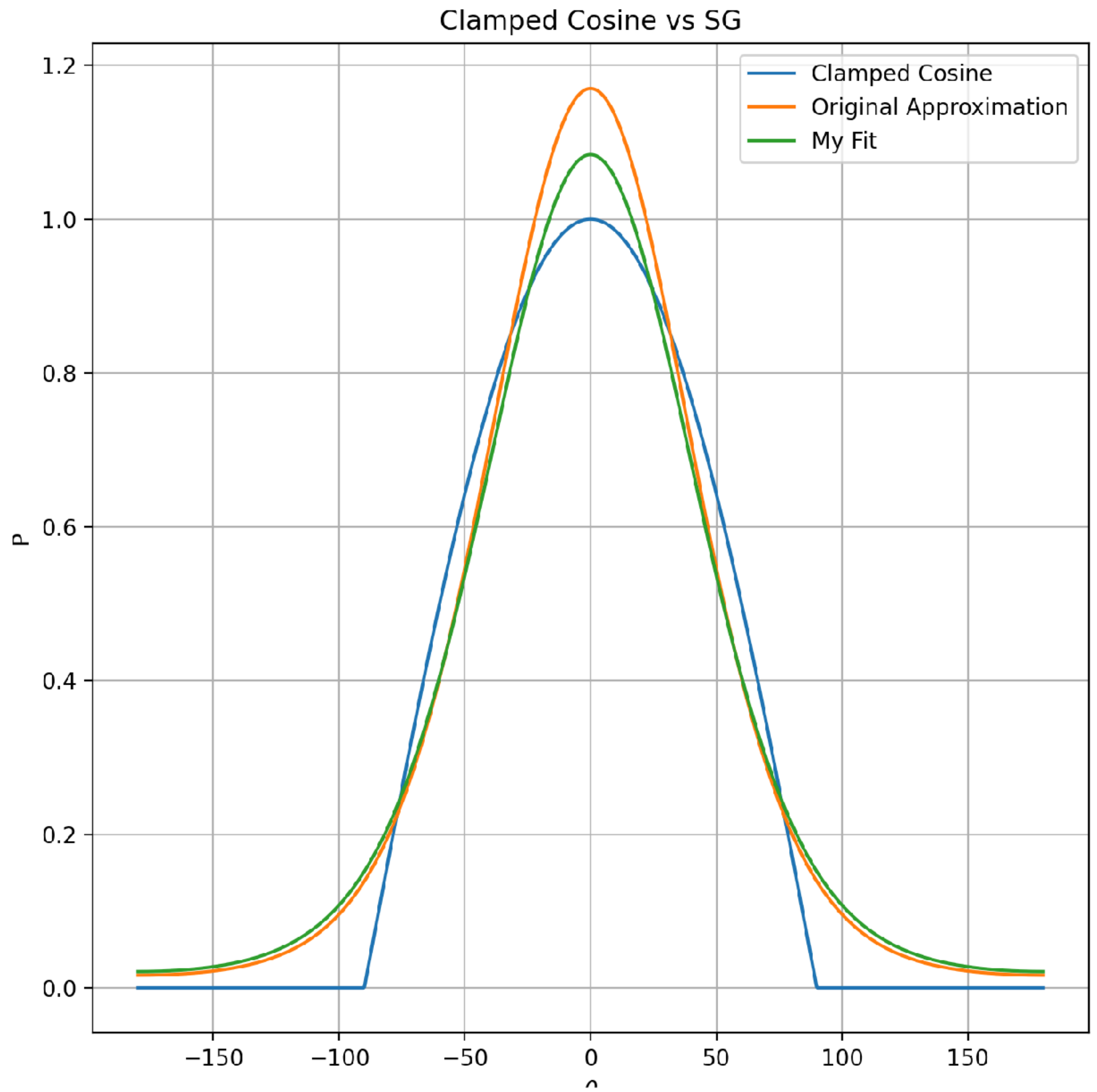
- Lambertian diffuse 是一个常数

$$R_d = \frac{K_d}{\pi} \int_{\Omega} L(\mathbf{i}) \max(0, \mathbf{i} \cdot \mathbf{n}) \mathrm{d}\omega$$

- 把光源和Cosine项分别使用两个 SG 来近似，然后利用 SG 内积的性质，求出解析解

Spherical Gaussians

- 假设我们可以使用一个 SG 来拟合一个面光源
- 我们还需要拟合 Clamped Cosine



Linearly Transformed Cosines

Real-Time Polygonal-Light Shading with Linearly Transformed Cosines

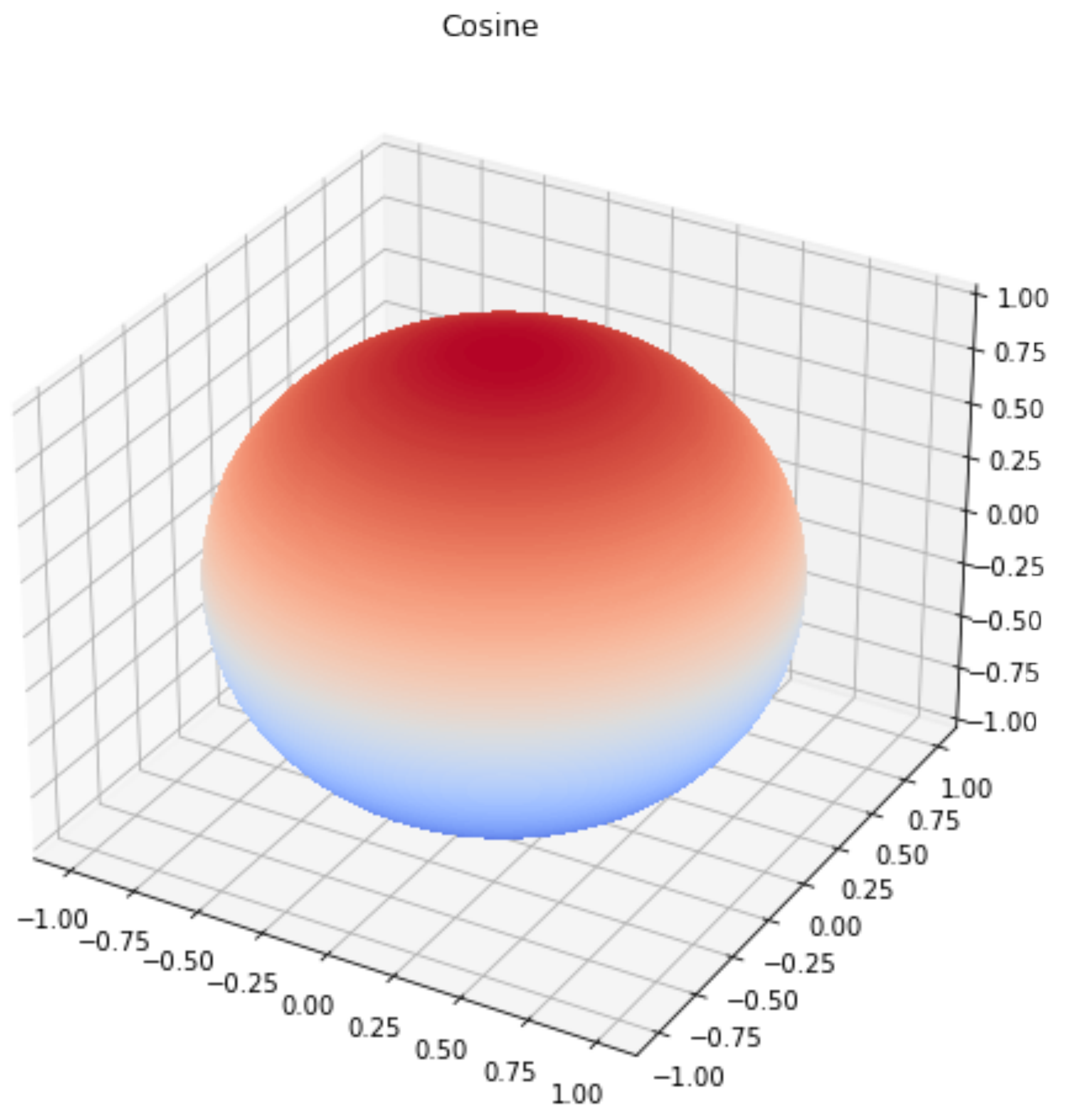
Eric Heitz, et al., SIGGRAPH 2016

Linearly Transformed Cosines

- Cosine 可以是一个定义在球面上的函数

$$D_o(\omega_o) = z_o$$

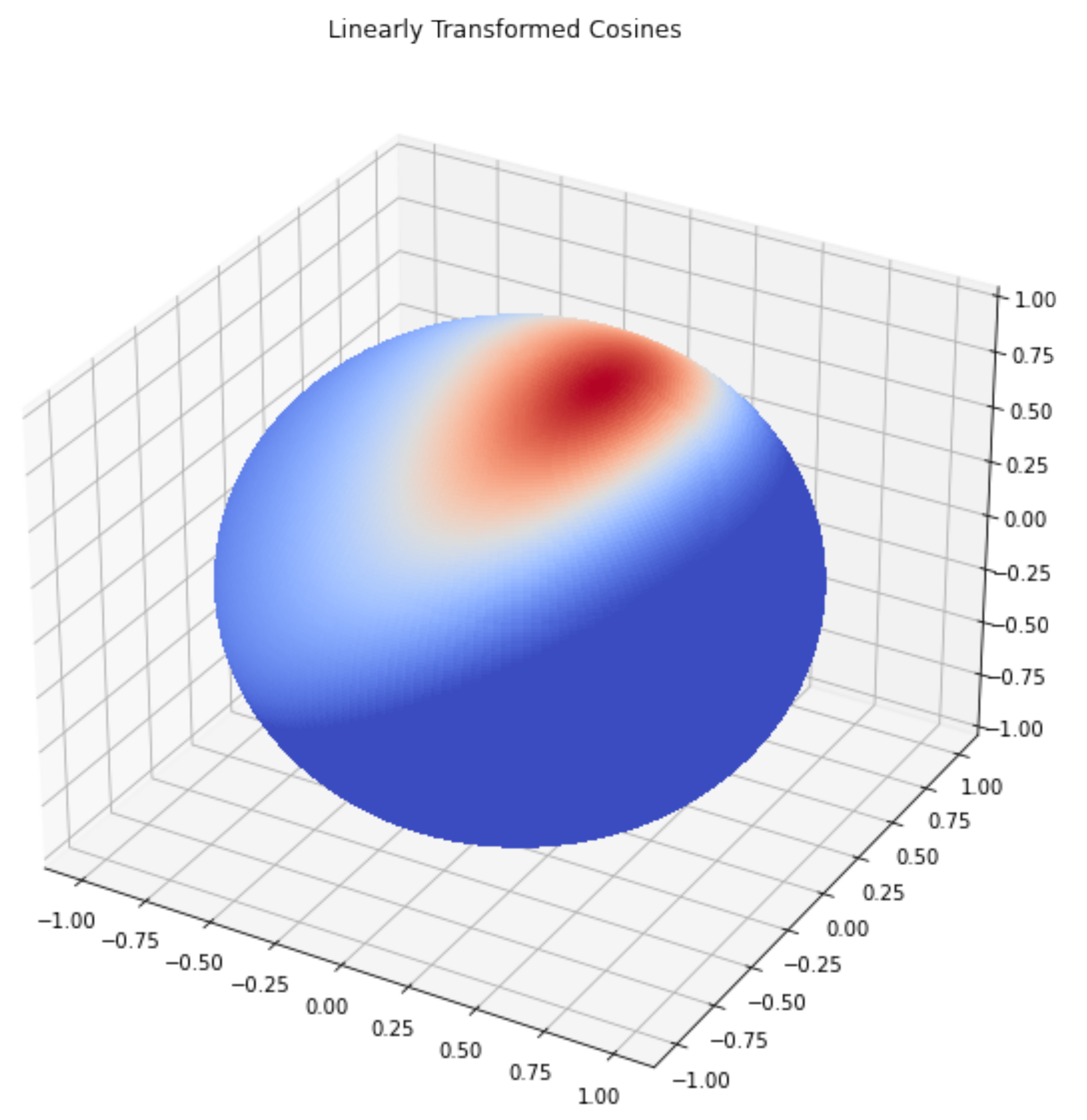
$$x = r\sin(\phi)\cos(\theta)$$
$$y = r\sin(\phi)\sin(\theta)$$
$$z = r\cos(\phi)$$



Linearly Transformed Cosines

- 对 Cosine 进行线性变换

$$D(\omega) = D_o\left(\frac{M^{-1}\omega}{\| M^{-1}\omega \|}\right)\frac{\left|M^{-1}\right|}{\| M^{-1}\omega \|^3}$$

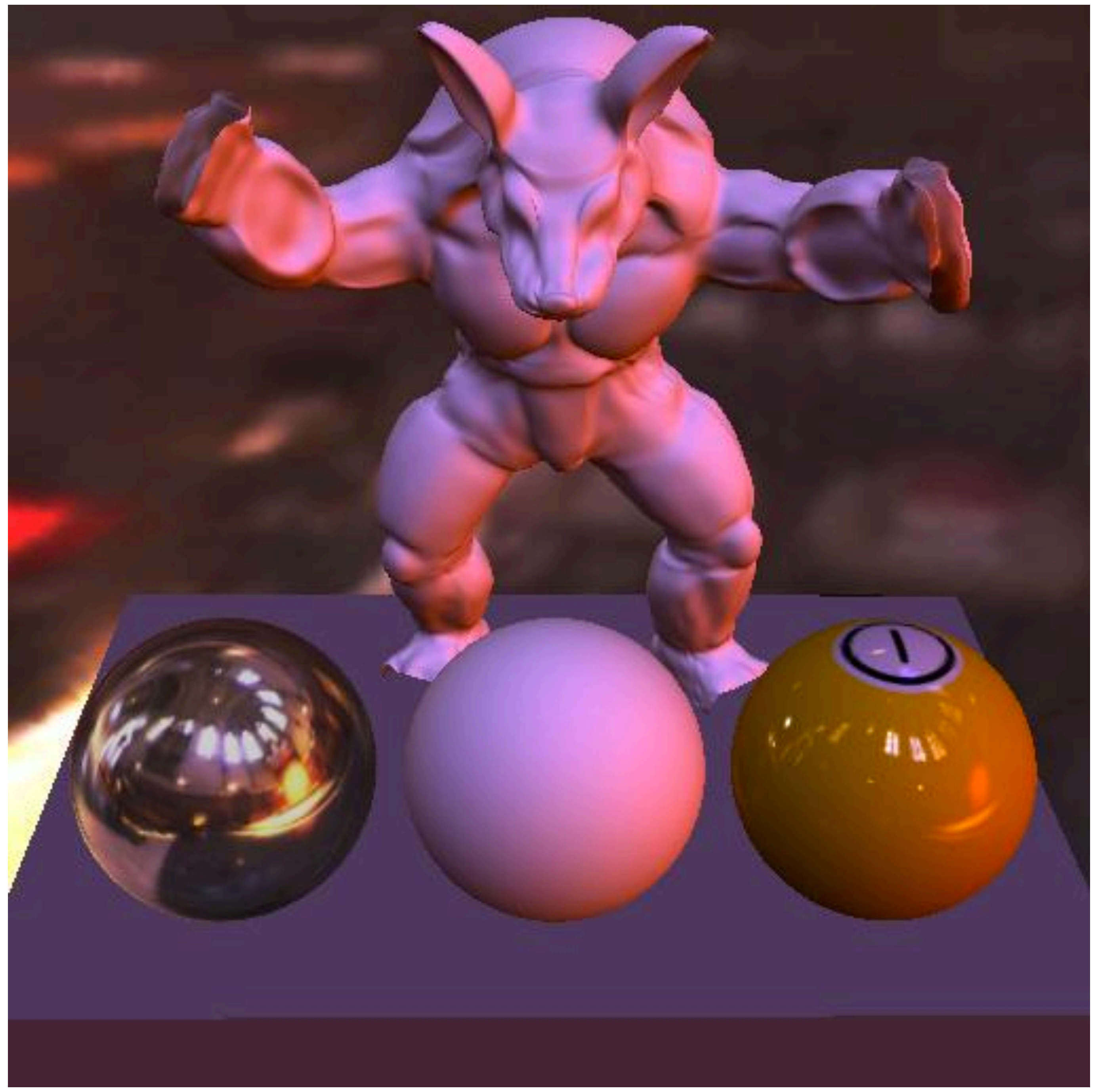


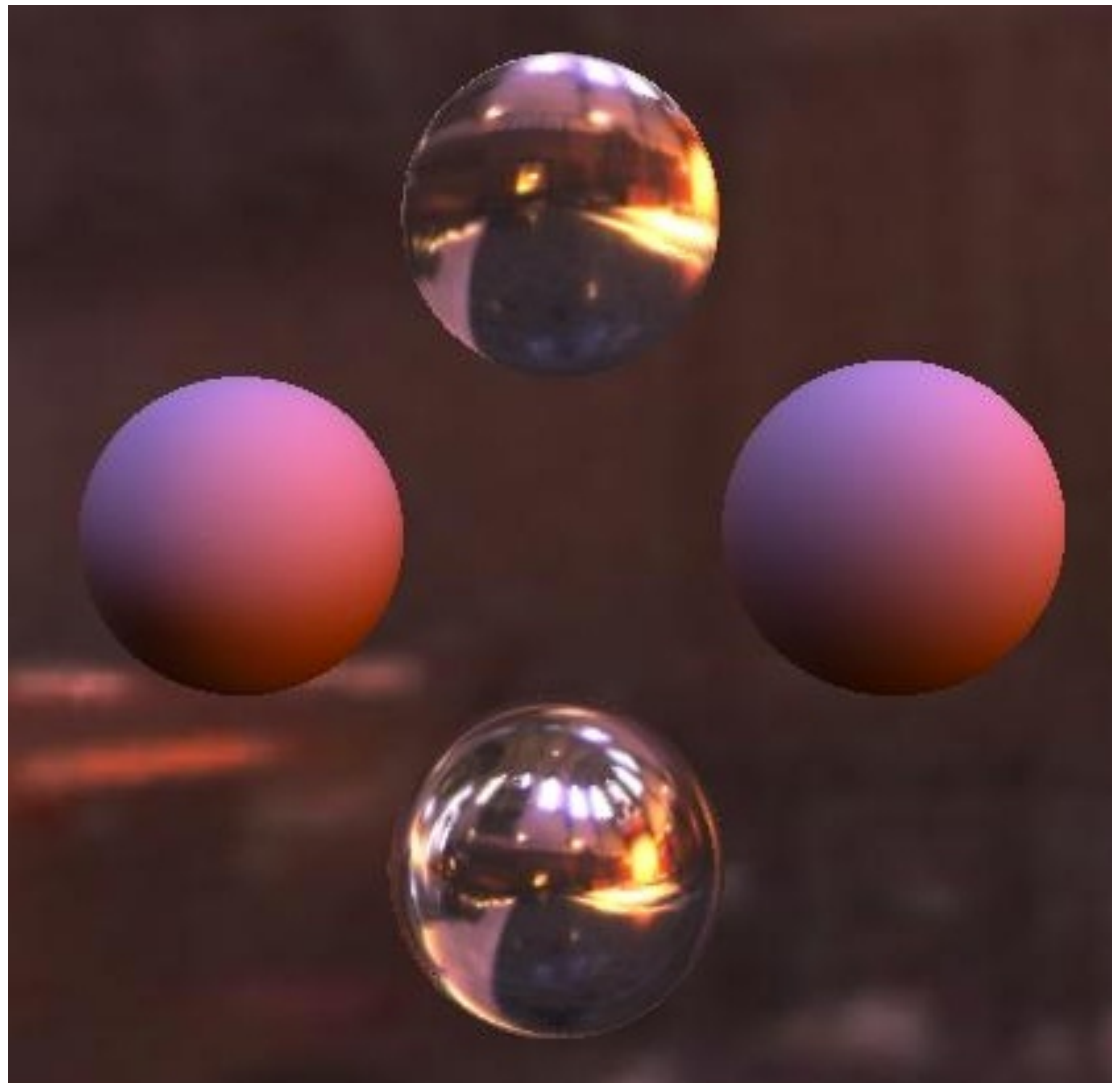
To be continued...

Spherical Harmonics

An Efficient Representation for Irradiance Environment Maps

Ravi Ramamoorthi, Pat Hanrahan, SIGGRAPH 2001

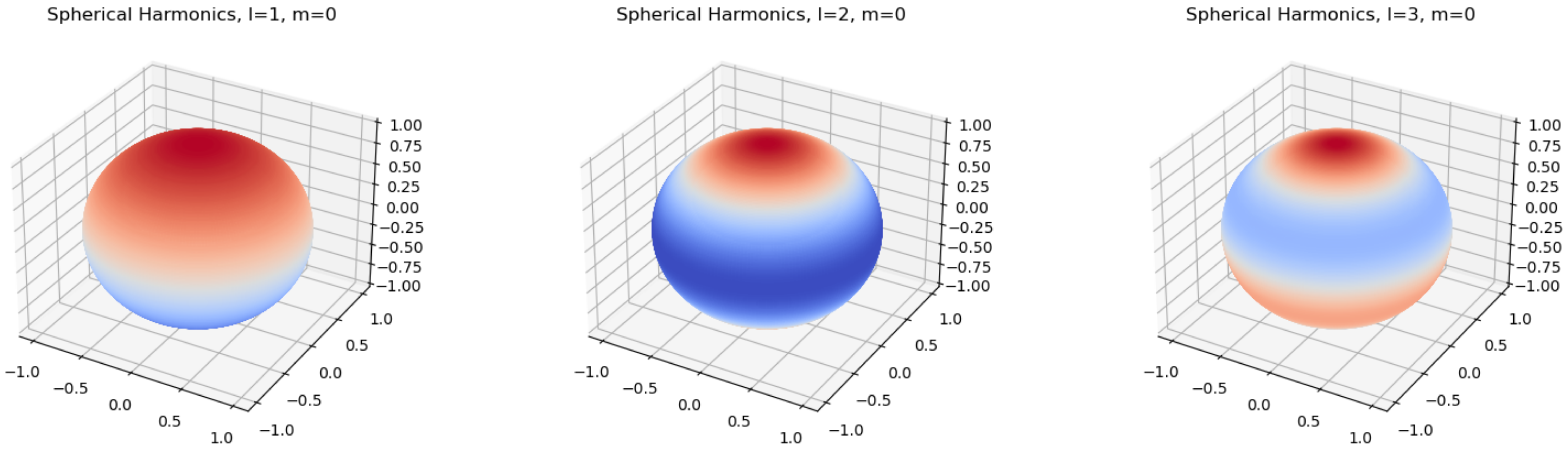




Spherical Harmonics

- SH一系列基函数，每个函数是定义在球面上的一个2D函数
- 用基函数的线性组合来拟合另外一个函数，通过投影即可计算出其系数

$$c_i = \int_{\Omega} f(\omega) B_i(\omega) \, \mathrm{d}\omega$$



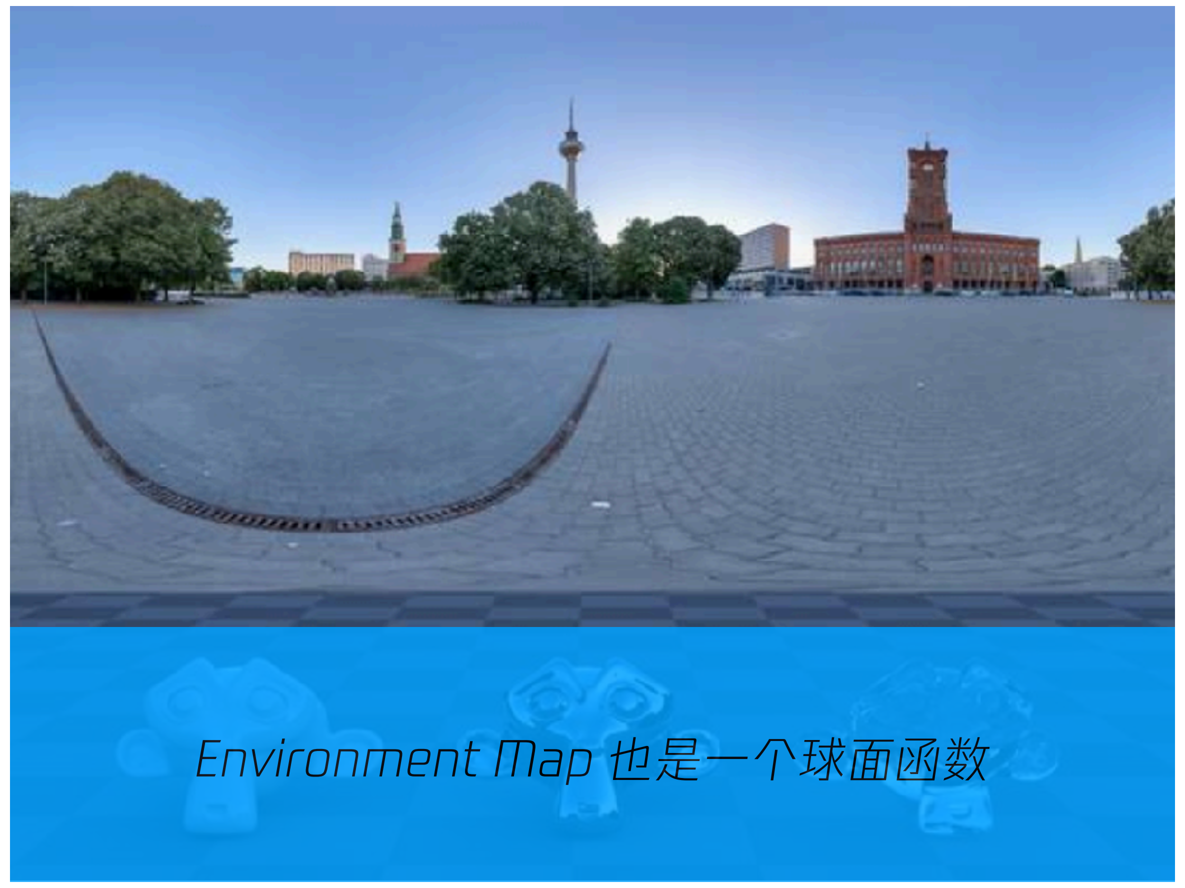
Spherical Harmonics

Precomputed radiance transfer for real-time rendering…

Peter-Pike Sloan et al., July 2002

$$L_r(\omega_o) = \int_\Omega L_i(\omega_i) f(\omega_i, \omega_o)(n \cdot \omega_i)\mathrm{d}\omega_i$$

使用 SH 拟合 Lighting



使用 SH 拟合其他部分

BRDF, Clamped Cosine, Visibility Term

Light Transport

也是一个球面函数

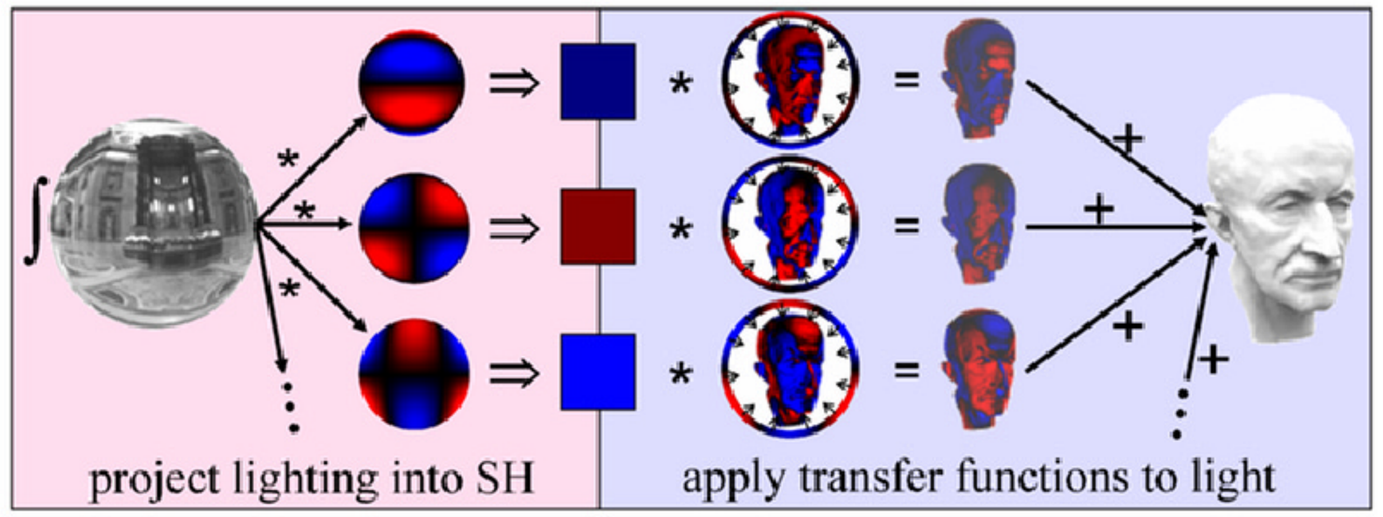
可以理解成为Shading Point 的性质，可以预计算

Spherical Harmonics

- SH 的正交性

$$\int_{\Omega} B_i(\mathbf{i}) \cdot B_j(\mathbf{i})\mathrm{d}\mathbf{i} = \mathbf{1} \quad (\mathbf{i} = \mathbf{j})$$

$$\int_{\Omega} B_i(\mathbf{i}) \cdot B_j(\mathbf{i})\mathrm{d}\mathbf{i} = \mathbf{0} \quad (\mathbf{i} \neq j)$$



diffuse surface self-transfer

- PRT 把积分转换成 SH 系数的点积

Split-Sum Approximation

实时渲染中的近似方法

Split-Sum Approximation

Real Shading in Unreal Engine 4

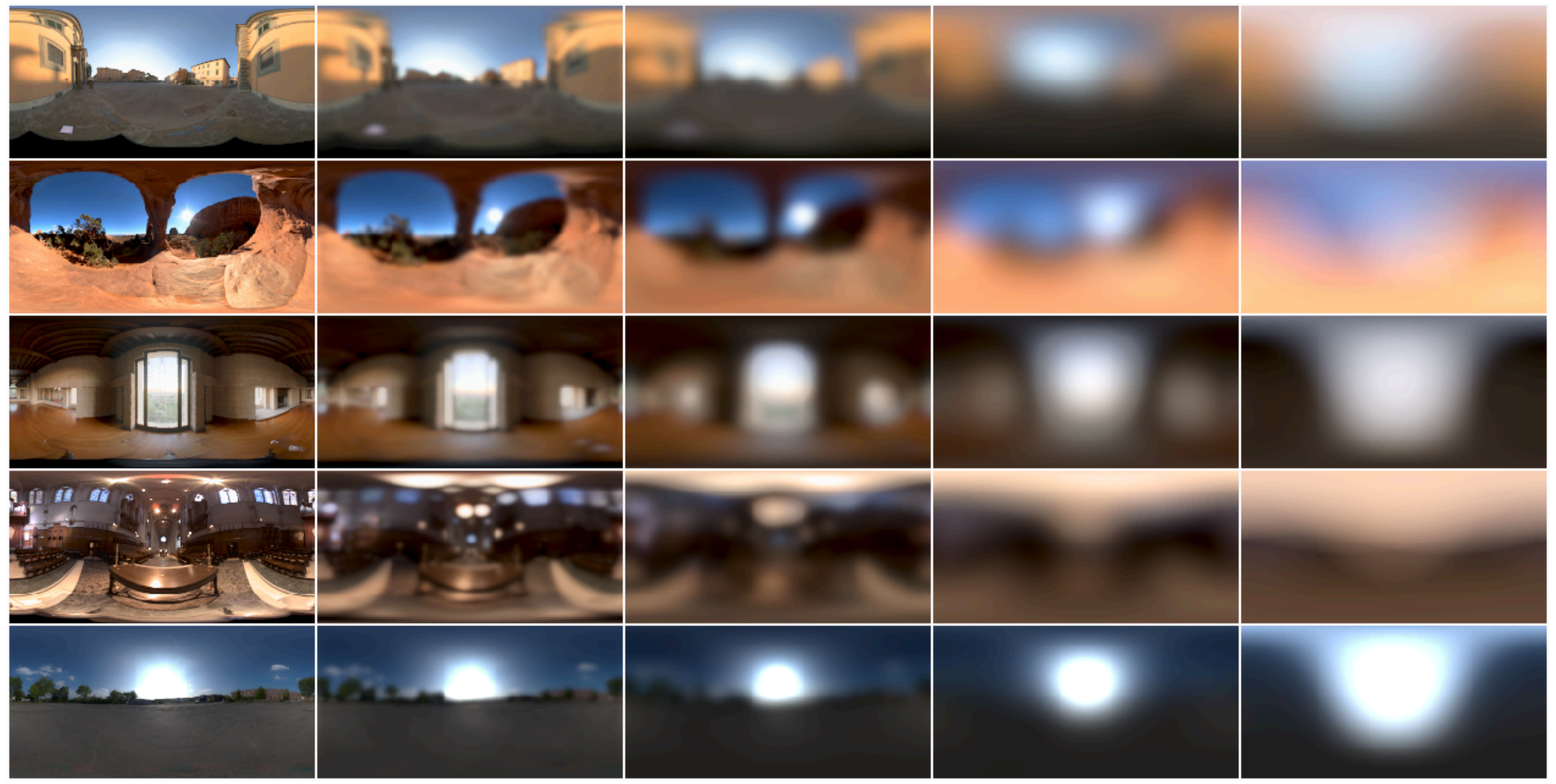
Brian Karis, SIGGRAPH 2013

$$\int_{\Omega} f(x)g(x)\mathrm{d}x \approx \frac{\int_{\Omega_G} f(x)\mathrm{d}x}{\int_{\Omega_G} \mathrm{d}x} \cdot \int_{\Omega} g(x)\mathrm{d}x$$

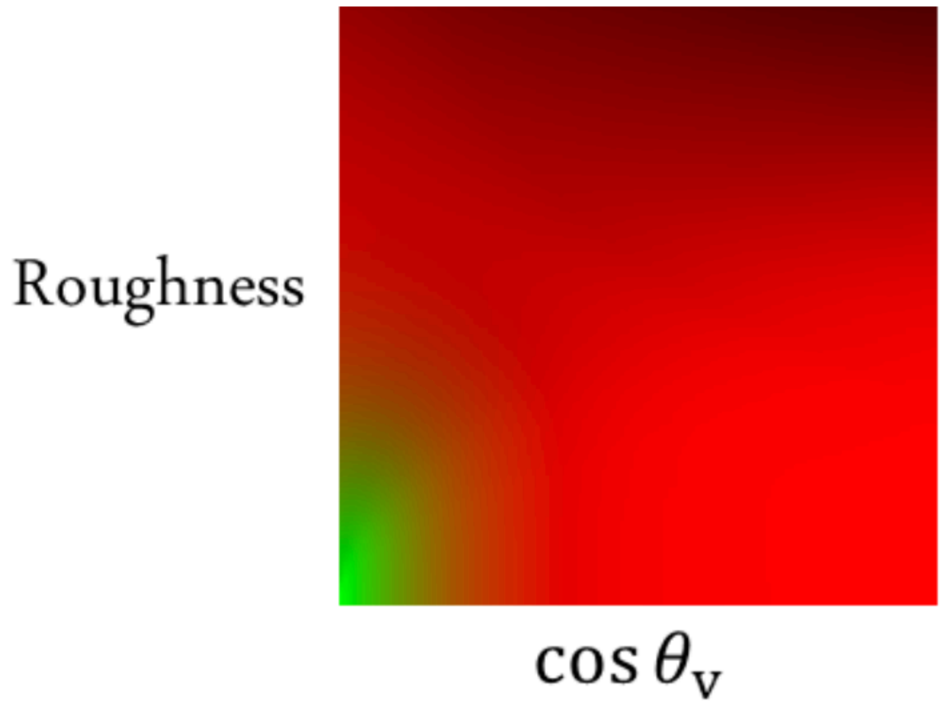
Split-Sum Approximation

$$L_o\left(p,\omega_o\right) = \int_{\Omega^+} L_i\left(p,\omega_i\right) f_r\left(p,\omega_i,\omega_o\right) \cos\theta_i \, \mathrm{d}\omega_i$$

$$L_o\left(p,\omega_o\right) \approx \frac{\int_{\Omega_+} L_i\left(p,\omega_i\right) \mathrm{d}\omega_i}{\int_{\Omega_+} \mathrm{d}\omega_i} \cdot \int_{\Omega^+} f_r\left(p,\omega_i,\omega_o\right) \cos\theta_i \, \mathrm{d}\omega_i$$



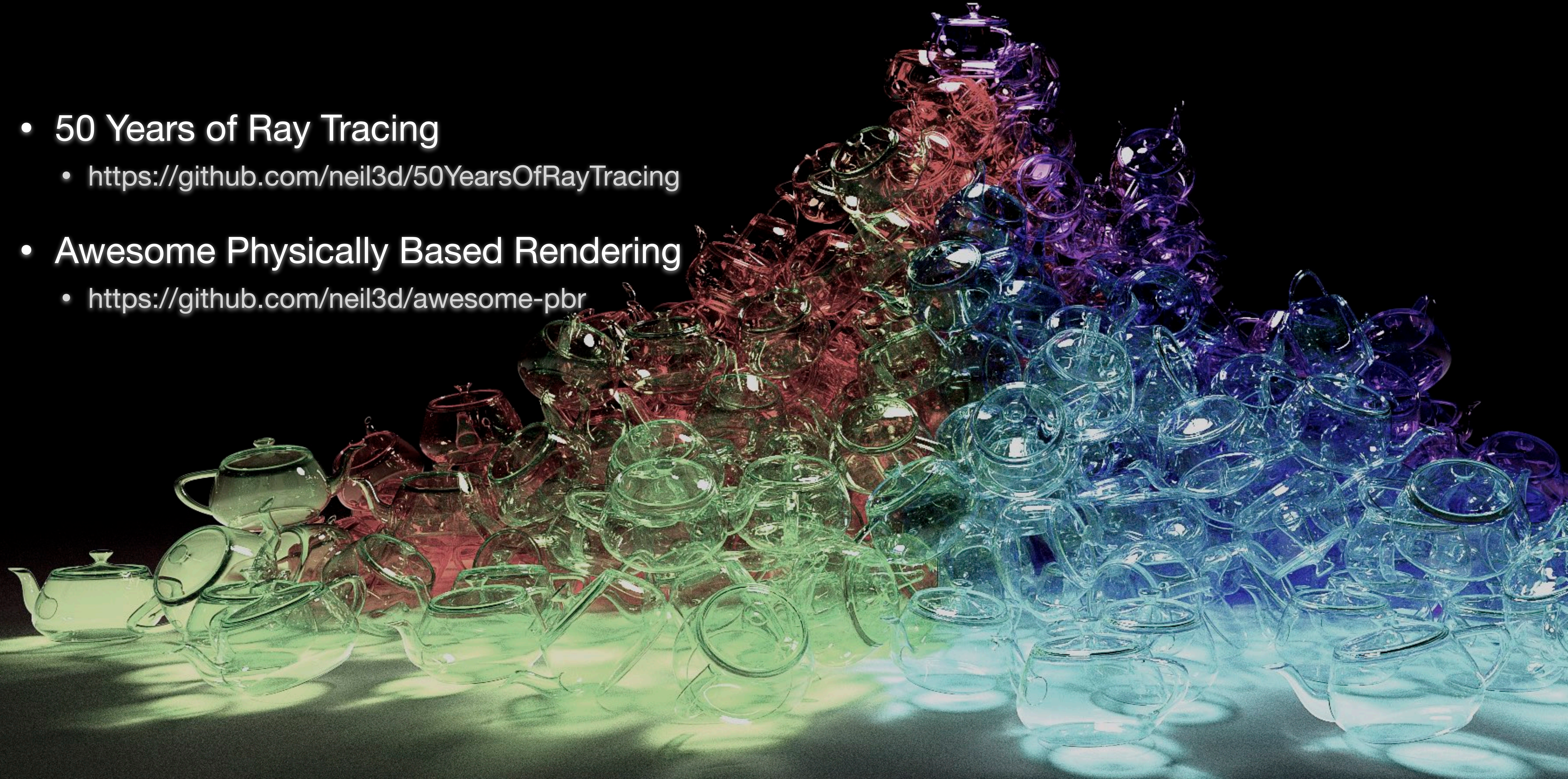
Pre-filtered Environment Maps



2D LUT

参考资料

- 50 Years of Ray Tracing
  - https://github.com/neil3d/50YearsOfRayTracing
- Awesome Physically Based Rendering
  - https://github.com/neil3d/awesome-pbr




Curiosity and passion determine success
Pat Hanrahan